The Ratchet Effect: A Learning Perspective*

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November 6, 2023

Abstract

We examine the ratchet effect under moral hazard and symmetric learning by worker and firm about new technology. Shirking increases the worker's future payoffs, since the firm overestimates job difficulty. High-powered incentives to deter shirking induce the agent to over-work, since he can quit if the firm thinks the job is too easy. With continuous effort choices, no deterministic interior effort is implementable. We provide conditions under which randomized effort is implementable, so that a profit-maximizing distribution over efforts exists.

Keywords: ratchet effect, moral hazard, learning, randomized effort.

JEL codes: D83, D86.

*Thanks for comments from Sunjoo Hwang, George Mailath, Meg Meyer, Alessandro Pavan, Caroline Thomas and numerous seminar audiences. We are indebted to three referees, the associate editor and the editor (Tilman Börgers) for their comments and for encouraging us to explore mixed strategies. Bhaskar thanks the National Science Foundation for its support via grant #SES1629055.
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1 Introduction

The ratchet effect arises when the principal/employer is uncertain about the technology and cannot make long term commitments. Previous work, notably Laffont and Tirole (1988), assumes that the agent understands the technology while the principal does not. In contrast, we assume that the technology is novel, and neither principal nor agent understand it. Our two-period model combines classical moral hazard with learning, where the principal can commit to spot contracts, but not to long-term ones. We have two main findings. First, we show an "impossibility result"—the principal cannot induce the agent to induce any deterministic interior effort in the first period. Second, we show, under more restrictive assumptions, that the principal can induce **some** non-degenerate distributions over interior effort, and that a profit-maximizing first-period contract exists.

We first explain our impossibility result, which is presented in section 3. Suppose that the principal seeks to induce a deterministic interior effort a^*. If the agent shirks and chooses $a < a^*$, and output y is publicly observed, the agent's beliefs regarding job difficulty will differ from that of the principal—the principal updates her beliefs assuming that a^* has been chosen while the agent knows that a was chosen. If the agent becomes more pessimistic than the principal, he incurs no loss – the job only pays him his reservation utility in equilibrium, and he can quit if he earns less. If he becomes more optimistic than the principal, he earns a rent. Under fairly general assumptions, we show that there must be some output level such that the agent is more optimistic than the principal when he shirks. Thus the agent raises his continuation value by shirking a little, and incentives have to be high powered in order to deter shirking. However, if the principal provides high powered incentives, this makes it profitable for the agent to over-work, i.e. to choose $a > a^*$, since by doing so, he increases his current payoff. This may cause the job to be less attractive tomorrow, but since the agent is always free to quit, he incurs no loss in consequence. Thus, at any interior effort level a^*, the left hand derivative of the agent's continuation value function is negative, while the right hand derivative is non-negative. This convex kink implies that the necessary first order conditions for implementing a^* can never be satisfied.

In Section 4 we ask if the principal can induce distributions over interior effort by providing moderate incentives.[1] Since the analysis is complex, we invoke more restrictive assumptions, including binary output, so that first period effort incentives can be summarized by the bonus for high output, Ω. A moderate bonus induces a unique continuation equilibrium where the agent randomizes over an interval of efforts $[\underline{a}_\Omega, \bar{a}_\Omega]$ according to a distribution G_Ω. Consequently, after each output realization y, the principal faces a second-period screening problem, where the agent's belief regarding the technology is distributed $F_{\Omega,y}$. This distribution is tightly related to

[1] That is, incentives are sufficient to induce positive effort, but insufficient to induce maximal effort.

the distribution of first period effort, G_Ω. If the agent chooses the highest equilibrium, effort level $\bar{a}_\Omega$, our more restrictive assumptions ensure that he is most pessimistic after both signals, and receives no informational rent. For any lower effort in the the support, his expected second period informational rent is exactly offset by the foregone first period payoff corresponding to the bonus Ω, so that the agent is indifferent between all efforts in $[\underline{a}_\Omega, \bar{a}_\Omega]$. Finally, we show that the principal's overall payoff is continuous in Ω, so that a profit-maximizing first-period contract exists.

The mixed strategy equilibrium provides a solution to the problem of the convex kink, via the choice of the distribution G_Ω, which affects informational rents. The largest equilibrium effort, $\bar{a}_\Omega$, maximizes the agent's first period payoff, and thus upward deviations are unprofitable, since they yield no future rents. The subtlety arises in ensuring that the left-hand derivative of the agent's expected continuation value is zero at $\bar{a}_\Omega$, which can be achieved if the principal induces zero effort from type $\bar{a}_\Omega$ after both signals. Intuitively, the agent's informational rents from more optimistic beliefs derive from the size of the second-period bonus, and vanish if this is set to zero. If the lowest equilibrium effort, $\underline{a}_\Omega$ is strictly positive, the loss in expected informational rent from marginal additional effort[2] exactly equals the gain in current payoff given the bonus. In other words, the expected continuation value function is now differentiable everywhere, including at the boundaries $\underline{a}_\Omega$ and $\bar{a}_\Omega$, thereby overcoming the impossibility that arose with deterministic effort.

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1.1 Related literature

The ratchet effect was first noted in the context of Soviet central planning (Berliner (1957), Weitzman (1980)), and arises in many contexts, including employer-worker relations in capitalist firms (Lazear (1986), Gibbons (1987), Carmichael and MacLeod (2000)) and regulation (Meyer and Vickers (1997), Freixas, Guesnerie and Tirole (1985)). It is a plausible explanation for delays in the introduction of new technology—see, for example, Atkin, Chaudhry, Chaudry, Khandelwal and Verhoogen (2017), who argue that it explains the reluctance of soccer ball producers in Pakistan to adopt a profitable innovation.

Previous formal work on the ratchet effect assumes that the worker has private information at the outset, and is most fully developed in Laffont and Tirole (1988), who assume a continuum of types. While they are not able to characterize the optimal contract[3], one general result is that the principal can never induce full separation, via **any** contract. When the maximal difference between types is small, pooling may be optimal, in which case first period effort will be excessive for less productive types,

[2]In a candidate pure effort equilibrium, this loss is zero, since the agent is left no rents.

[3]To quote Laffont and Tirole (1993, p. 376), "The extremely complex nature of equilibria makes it hard to characterize the optimal incentive scheme."

and insufficient for productive ones.[4] Laffont and Tirole (1993) examine the case of binary types and show that full separation may be not possible since it is vulnerable to the "take the money and run" strategy, where the low type mimics the high type and quits in the second period. This is somewhat similar to our pure strategy non-existence result. Their general conclusion is that a general characterization of optimal incentive schemes without commitment is very difficult, even with binary types.

The closest learning antecedent to this paper appears in the book by Milgrom and Roberts (1992). They assume symmetric, normally distributed uncertainty, and that output is the sum of technology, effort and noise. They argue that that high-powered incentives are needed to incentivize effort; however, they fail to take into account the possibility that the worker overworks and quits.[5]

Our results are related to papers that find that pure strategy equilibria do not exist in one-period models without commitment. Fudenberg and Tirole (1990) examine classical moral hazard with a risk averse agent, where the principal cannot commit not to renegotiate the contract at the interim stage, after effort is chosen but before output is realized. They show that a pure strategy equilibrium does not exist, and characterize mixed strategy equilibria. Bhaskar and Roketskiy (2021) analyze dynamic non-linear pricing with non-separable preferences and characterize the consequent mixed strategy equilibrium.

Gul (2001) and González (2004) study the hold up problem when the seller's investment is unobserved, and show that equilibrium must be in mixed strategies. Gul (2001) assumes Coasian bargaining—the buyer cannot commit not to revise his offer. He shows that as bargaining delay vanishes, the seller appropriates all the surplus from his investment, so that investment becomes efficient. González (2004) assumes that the buyer can commit, after investment is sunk, to a menu. This assumption is closer to ours. He shows that there is a unique mixed equilibrium, where the unobservability of investment and consequent informational rents partially alleviate under-investment. Solving for a mixed strategy equilibrium is more complex in our context than in the hold-up case, since random effort gives rise to several second period screening problems, one for each output realization. Consequently, the agent's indifference condition involves the expected informational rent, complicating its solution. Furthermore, in our case the second period problem combines adverse selection and moral hazard, whereas only adverse selection is present in the case of the hold-up problem. Finally, whereas there is no initial contract in Gonzalez's version of the hold-up problem, in our case, the principal affects effort incentives by her first period contract, and different bonus levels induce different equilibrium distributions of effort, and the principal must optimally choose her bonus.

[4] Malcomson (2016) shows that the no full-separation result also obtains in a relational contracting setting, where the principal need not have all the bargaining power, as long as continuation play following full separation is efficient.

[5] Their analysis could be rationalized by assuming that the employer leaves rents to the worker, so that the participation constraint does not bind, as we did in a previous version of this paper.

We now set out the model. Section 3 examines determinstic effort, while section 4 considers random effort.

2 The model

Our model combines moral hazard with uncertainty regarding job difficulty. There are two states of the world $\omega \in \{B, G\}$, with the job being good (easy) in G, and bad (hard) in B, with $\lambda \in (0, 1)$ denoting the common prior that $\omega = G$. The uncertainty concerns how difficult it is to succeed on **this** job. Importantly, learning does **not** affect the outside option of the agent, which is fixed, and normalized to 0.

We assume that the principal and agent interact for two periods—two periods suffice to make the main points of our paper. A key assumption is the absence of inter-temporal commitments—neither the principal nor the agent can commit in period one regarding the contract in period two. This implies that payments must satisfy incentive compatibility and individual rationality period by period.

The agent chooses effort $a \in [0, 1]$ in the first period and effort $b \in [0, 1]$ in the second. The costs of effort are $c_1(a)$ and $c_2(b)$ respectively; both functions are increasing, strictly convex and differentiable. The agent's flow payoff in period t from a utility level u (resulting from a wage payment by the principal), and effort level $e \in [0, 1]$ is $u - c_t(e)$. Both the agent and the principal seek to maximize the sum of their respective flow payoffs.[6]

At the end of period, the agent learns knowing his own effort choice and a realized public signal, $y \in Y$, where $Y := \{y^1, y^2, \ldots, y^K\}$ is a finite set of signals. The principal learns observing only the signal, since the agent's effort is private.

For our main result on pure strategy equilibria (deterministic effort), we do not need to invoke any assumption on the risk preferences of the principal or the agent. It suffices to assume that the two parties have strictly opposed preferences over monetary transfers, so that the utility cost to the principal of providing agent utility u is strictly increasing in u. That is, the principal's flow utility from an output level y and a utility payment u is strictly decreasing in u and strictly increasing in y. We assume that feasible utility payments are unrestricted, i.e. there is no limited liability.

A spot contract specifies a utility payment as a function of the realized signal, and is written as $\mathbf{u} := (u^1, u^2, \ldots, u^K)$, where u^k is the gross utility the agent will receive after signal y^k. In each period, the principal makes a take-it-or-leave-it offer of a spot contract to the agent. If the agent refuses, the relationship is dissolved and the game ends.

The probability of signal y^k at action e and state $\omega \in \{B, G\}$ is $p^k_{e\omega}$. Consider first the extremal efforts, $e \in \{0, 1\}$. We will assume that a "high" signal is both a signal of the good state and of high effort. We capture this by the following assumption.

[6]It is straightforward to allow for discounting, and the discount factors of two players need not be the same.

Assumption 1.

1. *There exists an **informative** signal, i.e.,* $\exists y^k \in Y$ *such that* $p^k_{0B} \neq p^k_{1G}$. *For any informative signal* $y^k \in Y$,

$$\min\left\{p^k_{0B}, p^k_{1G}\right\} \leq p^k_{0G} \leq \max\left\{p^k_{0B}, p^k_{1G}\right\},$$

and

$$\min\left\{p^k_{0B}, p^k_{1G}\right\} \leq p^k_{1B} \leq \max\left\{p^k_{0B}, p^k_{1G}\right\}.$$

2. *Signals have full support:* $p^k_{e\omega} > 0$ *for all* k, e, ω.

Partition the set of signals into a set of "high" signals Y^H, "low" signals Y^L, and neutral $Y \setminus (Y^H \cup Y^L)$ by defining

$$y^k \in Y^H \iff p^k_{1G} > p^k_{0B}$$

and

$$y^k \in Y^L \iff p^k_{1G} < p^k_{0B}.$$

Let μ denote the probability assigned to state G. The probability of signal y^k at effort level $e \in \{0, 1\}$ and belief μ is $p^k_{e\mu} = \mu p^k_{eG} + (1-\mu) p^k_{eB}$. Assumption 1 implies that for any informative signal y^k and any interior μ

$$y^k \in Y^H \iff p^k_{1G} \geq p^k_{1B}, p^k_{0G} \geq p^k_{0B},^7 \iff p^k_{1\mu} > p^k_{0\mu}$$

and

$$y^k \in Y^L \iff p^k_{1G} \leq p^k_{1B}, p^k_{0G} \leq p^k_{0B}, \iff p^k_{1\mu} < p^k_{0\mu}.$$

In other words, we can partition the signal space into high, low and neutral signals. High (resp. low) signals arise with higher (resp. lower) probability when either the agent exerts effort **or** the state is good. With binary signals (taking values H or L), where $p^H_{1G} > p^H_{0B}$, this assumption requires that p^H_{0G} and p^H_{1B} both belong to the interval $[p^H_{1G}, p^H_{0B}]$.

Our second assumption extends the information structure to all effort levels in $[0, 1]$. With a continuum of effort levels, we need to employ the first-order approach to solve for the optimal contract, even in the static case. We therefore assume the Hart and Holmström (1987) sufficient conditions for the validity of this approach, and our assumption is an adaptation of their conditions.

Assumption 2. *For any* $y^k \in Y$, *and any* $\omega \in \{G, B\}$, $p^k_{e\omega} = e p^k_{1\omega} + (1-e) p^k_{0\omega}$.[8]

[7] Assumption 1 implies that at least one of these weak inequalities, and those in next statement below, is strict.

[8] Hart and Holmström (1987) assume a linear cost of effort and that the probability of y^k is a convex combination of two distributions, a "good" one and a "bad". They assume that the weight on the good distribution is an increasing and concave function of effort. To see that our parameterization is equivalent to theirs, define a new effort variable, $c(e)$. This gives linear costs and a concave weighting function. This assumption is a special case of the spanning condition of Grossman and Hart (1983).

We study the dynamic game induced by this contracting problem, and solve for perfect Bayesian equilibria that satisfy sequential rationality, with beliefs given by Bayes rule.[9] Although differences in beliefs between the principal and the agent will play an important part in the analysis, we do not have to specify beliefs off the equilibrium path for the uninformed party (the principal). Since effort choice by the agent is private and public signals have full support, the principal does not see an out of equilibrium action, except when the game ends by the agent refusing the contract (at which point, beliefs are moot). Deviations by the uninformed party (the principal) have no implications for beliefs.

3 Equilibria with deterministic effort

In this section, we focus on pure strategy equilibria, where the effort choice by the agent in period one is deterministic. If the agent chooses effort a^* at $t = 1$, and output y^k is realized, then the common belief of the principal and agent at $t = 2$ is denoted by $\mu^k_{a^*}$. Because the agent plays a pure strategy, the principal's second order beliefs are degenerate. Sequential rationality implies that the principal offers a profit maximizing contract at $t = 2$. We assume that the project is profitable at all beliefs at $t = 2$, so that the principal always induces the agent to participate.

Suppose that after observing the results of the first period, the principal and the agent agree on how difficult the task is. Let μ denote a common belief of the principal and the agent at the beginning of the second period. Effort $\hat{b}$ is **implementable** at $t = 2$ if there exists a spot contract such that $\hat{b}$ is optimal for the agent under belief μ. It is immediate that in the final period, for any public belief μ, every effort $b \in [0, 1]$ is implementable. The agent's individual rationality constraint must bind in an optimal contract—otherwise, the principal can profit by reducing utility payments after each output signal uniformly by a small ϵ, without affecting the agent's effort incentives.

3.1 The agent's continuation value

We shall assume that for any μ, the effort induced by the principal, $\hat{b}(\mu)$, is nonzero—this assumption is a mild one if $c_2'(0) = 0$. We now analyze the agent's payoff in the final period when his belief is π and differs from the principal's belief μ. Recall that $\mathbf{u}_\mu$ denotes the optimal contract offered by the principal when her belief is μ. Let $\hat{V}(\pi, \mu) = \max_b (\mathbf{p}_{b\pi} \cdot \mathbf{u}_\mu - c_2(b))$ denote the payoff to the agent, conditional on accepting the job and choosing effort optimally.

$\hat{V}$ is computed under the distribution $\mathbf{p}_{b\pi}$, reflecting the fact that the agent has the correct beliefs, since he knows his actual effort choice at $t = 1$. Since the agent will quit when he gets less than his outside option, let $V(\pi, \mu) = \max\{\hat{V}(\pi, \mu), 0\}$ denote

[9]Sequential rationality ensures that the agent accepts any contract that offers him at least his reservation utility, thereby ruling out incredible threats.

his payoff given optimal participation. $V(\mu,\mu)=0$ when $\pi=\mu$ since the agent's participation constraint binds under the optimal contract if he has the same beliefs. The following lemma summarizes the relevant properties of V, the most important property being that the agent gets rents if $\pi>\mu$. Intuitively, since the second period contract must provide incentives for positive effort, it must reward high signals. But under Assumption 1, this also rewards more optimistic beliefs.

Lemma 1. *$V(\pi,\mu)>0$ if and only if $\pi>\mu$. $\hat{V}(\pi,\mu)$ is differentiable and $V(\pi,\mu)$ is convex in π.*

Proof. See Appendix A. □

Suppose that the principal seeks to induce effort level a^* at $t=1$, when both parties have common prior beliefs λ. If the agent deviates and chooses a different from a^*, then the principal and agent will have different second period beliefs after output y^k. The principal will have belief $\mu^k_{a^*}$, while the agent will have belief π^k_a. The expected second period continuation value of the agent from choosing a when the principal induces a^* equals

$$W(a,a^*)=\sum_{k=1}^{K}p^k_{a\lambda}V(\pi^k_a,\mu^k_{a^*}).$$

Each term under the summation sign is non-negative, since $V(\pi^k_a,\mu^k_{a^*})\geq 0$, given that the agent can always quit when $\pi^k_a<\mu^k_{a^*}$. Thus $W(a,a^*)$ is strictly positive as long as there is some y^k such that $\pi^k_a>\mu^k_{a^*}$.

Consider downward deviations by the agent to $a<a^*$. There will be some set of signals where the agent is more optimistic than the principal, and another set where where he is more pessimistic. The following lemma shows that signals can be partitioned into these sets independently of the precise magnitudes of a and a^*. For $a\in\{0,1\}$, let the likelihood ratios on states given effort a be denoted by $\ell^k_a:=\frac{p^k_{aG}}{p^k_{aB}}$. Define:

$$Y^D:=\{y^k\in Y:\ell^k_0>\ell^k_1\}.$$
$$Y^U:=\{y^k\in Y:\ell^k_0<\ell^k_1\}.$$

Lemma 2. *For any efforts $a,a^*$with $a<a^*$: $\pi^k_a>\mu^k_{a^*}$ if $y^k\in Y^D,\pi^k_a<\mu^k_{a^*}$ if $y^k\in Y^U$ and $\pi^k_a=\mu^k_{a^*}$ if $y^k\in Y\setminus(Y^U\cup Y^D)$.*

Proof. See Appendix A. □

The next lemma is key for our results, since it shows that Y^D is non-empty—there is at least one signal such that the agent is more optimistic than the principal when he shirks. We prove a more general result, that the agent is on average more optimistic than the principal, since it is of independent interest, for conceptual reasons, as well as practically. The proof is also of independent economic interest—the result follows from the martingale property of beliefs and Assumption 1. Let $\mathbf{E}_{0,\lambda}(\pi_0^k)$ denote the expectation of the agent's belief, given prior λ and effort 0, and let $\mathbf{E}_{0,\lambda}(\mu_1^k)$ the expectation of the principal's "false" belief, given that the principal believes that the agent is choosing $a = 1$, while in fact he is choosing $a = 0$.[10]

Lemma 3. $\mathbf{E}_{0,\lambda}(\pi_0^k) > \mathbf{E}_{0,\lambda}(\mu_1^k)$, *so that* Y^D *is non-empty.*

Proof. Let $\mathbf{E}_{1,\lambda}(\mu_1^k)$ denote the expectation of the belief of the principal when she correctly conjectures that the agent is choosing $a = 1$. From the martingale property of beliefs, $\mathbf{E}_{0,\lambda}(\pi_0^k) = \mathbf{E}_{1,\lambda}(\mu_1^k) = \lambda$, i.e.

$$\sum_{k=1}^{K} p_{0\lambda}^k \pi_0^k = \sum_{k=1}^{K} p_{1\lambda}^k \mu_1^k.$$

Subtract $\sum_{k=1}^{K} p_{0\lambda}^k \mu_1^k$ from both sides to get

$$\sum_{k=1}^{K} p_{0\lambda}^k (\pi_0^k - \mu_1^k) = \sum_{k=1}^{K} (p_{1\lambda}^k - p_{0\lambda}^k)\mu_1^k.$$

Observe that $\lambda \sum_Y (p_{1\mu}^k - p_{0\mu}^k) = 0$, since the sum of the difference between two probability distributions is zero. Consequently,

$$\sum_{k=1}^{K} p_{0\lambda}^k (\pi_0^k - \mu_1^k) = \sum_{k=1}^{K} (p_{1\lambda}^k - p_{0\lambda}^k)(\mu_1^k - \lambda).$$

Under Assumption 1, for any k, $(p_{1\lambda}^k - p_{0\lambda}^k)$ has the same sign as $(\mu_1^k - \lambda)$—i.e. a signal that has higher probability under high effort is also informative of the job being easier. Since there is some informative signal, we conclude that $\sum_{k=1}^{K} p_{0\lambda}^k (\pi_0^k - \mu_1^k) > 0$, i.e. the expectation of the difference in beliefs under the experiment $a = 0$ is strictly positive. Thus there must be some signal y^k such that $\pi_0^k > \mu_1^k$.

□

We have shown that the expectation of the "false belief" held by the principal, μ_1^k, that is induced when the agent performs the experiment $a = 0$, is strictly smaller than the expectation of the true belief π_0^k. Thus there must be some signal realization for which $\pi_0^k > \mu_1^k$. This immediately proves that the agent can increase his continuation value by deviating to low effort.

We illustrate this key lemma with three binary examples depicted in Table 1. Output is either high H or low L. Let p and q denote the probabilities of H given

[10] That is, the principal updates assuming that effort 1 has been chosen, but the distribution on output signals is given by effort 0 and the prior λ.

	$e=1$	$e=0$
$\omega=G$	p	q
$\omega=B$	q	q

(a)

	$e=1$	$e=0$
$\omega=G$	p	p
$\omega=B$	p	q

(b)

	$e=1$	$e=0$
$\omega=G$	p	$\frac{p+q}{2}$
$\omega=B$	$\frac{p+q}{2}$	q

(c)

Table 1: Binary signals: $\Pr(H|e,\omega)$

effort 1 in state G and effort 0 in state B respectively, with $p > q$. In Table 1a, effort is only productive in the good state. Since zero effort is uninformative, high effort is a more informative experiment than low effort. If the principal seeks to induce $a^* > 0$, and the agent shirks, choosing $a < a^*$, then he becomes more pessimistic than the principal after H, but more optimistic than the principal after L. Consequently, he will quit in the second period after success (output H), but earn informational rents after failure (output L). Next, consider Table 1b, where effort is only productive in the bad state. Since effort 1 is uninformative, low effort is a more informative experiment than high effort. If the principal induces $a^* > 0$, and the agent chooses $a < a^*$, then he becomes more pessimistic than the principal after output L, but more optimistic than the principal after output H. The agent will quit in the second period after failure (L), but earn informational rents after he succeeds (H). These two examples also illustrate that the agent earns informational rents not by choosing a more informative experiment, but rather by shirking. Finally, in Table 1c, effort is equally productive in the two states, If the agent shirks, he is more optimistic than the principal after both signals. Consequently, after shirking, he stays on the job and earns informational rents after both first period output realizations. This final example is similar to the linear additive structure in Milgrom and Roberts (1992). It also satisfies the more restrictive informational assumptions we invoke while analyzing mixed strategies in Section 4.

Lemma 3 follows from Assumption 1. Thus the ratchet effect obtains under a fairly general information structure—most existing work assumes either binary or normal signals. Assumption 1 plays a similar role in Bhaskar and Mailath (2019), which examines in the long run consequences of belief manipulation.

In the light of Lemma 2 we may re-write $W(a,a^*)$ as[11]

$$W(a,a^*) = \begin{cases} \sum_{y^k \in Y^D} p^k_{a\lambda} V(\pi^k_e, \mu^k_{a^*}) \text{ if } a < a^* \\ \sum_{y^k \in Y^U} p^k_{a\lambda} V(\pi^k_a, \mu^k_{a^*}) \text{ if } a > a^*. \end{cases}$$

[11]Note that $W(a^*,a^*) = 0$.

3.2 Non-existence of pure effort equilibria

The overall payoff of the agent from effort choice a in the first period, given a contract $\mathbf{u}$ that seeks to induce effort level a^*, is given by

$$\mathcal{U}(a; a^*, \mathbf{u}) = \mathbf{u} \cdot \mathbf{p}_{a\lambda} - c_1(a) + W(a, a^*).$$

The following theorem and its generalization (Theorem 2 in Section 3.3) are the main negative results of this paper.

Theorem 1. *Assume that the principal optimally induces non-zero effort at $t = 2$ at every belief. If $a^* \in (0, 1)$, then a^* is **not** implementable at $t = 1$. The extremal efforts 0 and 1 are implementable.*

Proof. We evaluate the left-hand and right-hand partial derivatives of $W(a, a^*)$ with respect to its first argument at $a = a^* \in (0, 1)$, and show that these are inconsistent with the first order conditions for implementing a^*.The left hand partial derivative is given by

$$W_1^-(a^*, a^*) = \sum_{k=1}^{K}(p_{1\lambda}^k - p_{0\lambda}^k)V(\pi_{a^*}^k, \mu_{a^*}^k) + \sum_{y^k \in Y^D} p_{a^*\lambda}^k V_\pi^+(\pi_{a^*}^k, \mu_{a^*}^k) \left.\frac{\partial \pi_a^k}{\partial a}\right|_{a=a^*}. \quad (1)$$

The first term is zero since $V(\mu, \mu)$ is constant, since $\sum_{k=1}^{K}(p_{1\lambda}^k - p_{0\lambda}^k) = 0$. By lemma 1, the right-hand derivative of V is the partial derivative of $\hat{V}$, with respect to its first argument, and thus

$$W_1^-(a^*, a^*) = \sum_{y^k \in Y^D} p_{a^*\lambda}^k \hat{V}_\pi(\pi_{a^*}^k, \mu_{a^*}^k) \left.\frac{\partial \pi_a^k}{\partial a}\right|_{a=a^*}. \quad (2)$$

The partial derivative of $\hat{V}$ equals $\left(\mathbf{p}_{\tilde{a}(\mu)G} - \mathbf{p}_{\tilde{a}(\mu)B}\right).\mathbf{u}_\mu > 0$. From equation (10), $\left.\frac{\partial \pi_a^k}{\partial a}\right|_{a=a^*}$ has the same sign as $\left[p_{1G}^k p_{0B}^k - p_{1B}^k p_{0G}^k\right]$, which is strictly negative if $y^k \in Y^D$. Since each term in the summation is strictly negative, $W_1^-(a^*, a^*) < 0$.The right-hand derivative, $W_1^+(a^*, a^*)$, is bounded below by zero, since $W(a, a^*) \geq 0$ and $W(a^*, a^*) = 0$. If there is uniform optimism, $W_1^+(a^*, a^*) = 0$, since Y^U is empty. If Y^U is non-empty, then similar arguments as for the left-hand derivative show that $W_1^+(a^*, a^*) > 0$.

The agent's current payoff at $t = 1$ is a smooth function of effort, given the differentiability of the cost of effort and expected utility. Thus the first order conditions for a^* to be optimal for the agent at $t = 1$ are:

$$\left(\mathbf{p}_{1\lambda} - \mathbf{p}_{0\lambda}\right).\mathbf{u} - c_1'(a^*) + W_1^-(a^*, a^*) \geq 0.$$

$$\left(\mathbf{p}_{1\lambda} - \mathbf{p}_{0\lambda}\right).\mathbf{u} - c_1'(a^*) + W_1^+(a^*, a^*) \leq 0.$$

Since $W_1^-(a^*, a^*) < W_1^+(a^*, a^*)$, the two conditions cannot be simultaneously satisfied, thereby proving the main part of theorem.

The extremal efforts, 0 and 1, can be implemented, since one has to only deter deviations in one direction. For implementing $a = 0$, if the signal structure satisfies uniform optimism, then a constant utility schedule implements it, and will also be optimal if neither party is risk loving. However, if the signal structure does not satisfy uniform optimism, then the agent may have to be punished for higher output levels—he may have an incentive to deviate upwards, since he will be more optimistic than the principal after some output realizations. It suffices to choose utility payments $\mathbf{u}$ such that $\mathbf{p}_{a\lambda} \cdot \mathbf{u} - c_1(a) + W(a, 0)$ is maximized at $a = 0$. If $c_1(\cdot)$ is sufficiently convex (to offset the convexity of the W function), then the first order condition suffices:

$$(\mathbf{p}_{1\lambda} - \mathbf{p}_{0\lambda}) . \mathbf{u} - c_1'(0) + W_1^+(0, 0) \leq 0.$$

Similarly, $a = 1$ can also be implemented, by choosing $\mathbf{u}$ so that $\mathbf{p}_{a\lambda} \cdot \mathbf{u} - c_1(a) + W(a, 1)$ is maximized at $a = 1$. □

Remark 1. *We have assumed that the principal finds it profitable to employ the agent in period* 2 *and to always induce non-zero effort. Neither assumption is required: it suffices that she finds it optimal to employ the agent and induce non-zero effort after* ***some*** *signal in* Y^D*, the set of signals where the agent is more optimistic after downward deviations.*

Remark 2. *This stark result arises from the fact that the agent's participation constraint binds, in equilibrium, in the second period. If the agent were to earn rents, then interior effort may be implementable. Rents can arise if the agent is subject to limited liability. They can also arise due to the agent having some bargaining power or due to the employer paying more than the outside option due to "fairness" considerations.*

The negative result in Theorem 1 is striking: no interior effort level can be implemented in the first period. The ratchet effect implies that the agent can raise his continuation value by shirking a little relative to a^*. To overcome this, incentives today must be high powered, so that a little shirking reduces the agent's current payoff. This implies that the agent can also increase his current payoff by over-working relative to a^*—this follows from the fact that current costs and benefits are smooth functions of effort. However, over-working cannot **reduce** the agent's continuation value relative to a^*, since the agent can always quit. In other words, the principal can deter downward deviations, but this makes upward deviations profitable. Thus high powered incentives cannot overcome the ratchet effect, contrary to the argument in Milgrom and Roberts (1992).

3.3 Generalizing the impossibility result

We now show that the impossibility result in Theorem 1 arises under very mild informational conditions. Specifically, it is true generically as long as signals depend upon the state and also depend upon effort. We replace Assumption 1 with the following:

Assumption 1*. *1. Signals have full support: $p^k_{e\omega} > 0$ for all k, ω and $e \in \{0,1\}$.*

2. There exists $k \in \{1, 2.., K\}$, such that $p^k_{1\omega} \neq p^k_{0\omega}$ for some $\omega \in \{G, B\}$ and $p^k_{eG} \neq p^k_{eB}$ for some $e \in \{0,1\}$.

Assumption 1* implies that there is an output realization $\tilde{y}$, such that if the agent's true effort differs from the effort the principal believes the agent exerted, the two have differing beliefs.

As before, we invoke Assumption 2 in to extend the information structure to all effort levels in $[0,1]$, so that $\forall \omega \in \{G, B\}, p^k_{e\omega} = ep^k_{1\omega} + (1-e)p^k_{0\omega}$.

In the final period, for any public belief $\mu \in [0,1]$, every effort $b \in [0,1]$ is implementable, and the optimal final contract after history can be found using the first-order approach. We assume that effort is productive,and that the principal wants to induce positive effort at any belief in the static problem. We also assume that the principal always finds it profitable to employ the agent. We now analyze the agent's payoff in the final period, $V(\pi, \mu)$, when his belief is π and differs from the principal's belief μ. Since the agent can always quit, $V(\pi, \mu) \geq 0$. We now show that generically, the agent will get an informational rent, i.e. $V(\pi, \mu) > 0$ either when $\pi > \mu$ or when $\pi < \mu$.

Recall that the informational structure $\mathbf{p} = (\mathbf{p}_{0G}, \mathbf{p}_{0B}, \mathbf{p}_{1G}, \mathbf{p}_{1B})$ is an element of $\Delta^{4(K-1)}$ and the vector of possible output realizations $\mathbf{y} = (y_1, .., y_K)$ lies in $\mathbb{R}^K$. We label the pair $(\mathbf{p}, \mathbf{y})$ the **parameters of the model**. Thus the parameters of the model live in $\Delta^{4(K-1)} \times \mathbb{R}^K$.

The following lemma is the main step in the proof of the theorem. All proofs for this sub-section can be found in Appendix B.

Lemma 4. *Supppose that informational Assumptions 1* and 2 are satisfied. For almost all parameters of the model, $V(\pi, \mu) > 0$ either when $\pi > \mu$ or when $\pi < \mu$.*

Proof. See Appendix B. □

We now examine how deviations by the agent from the effort level induced by the principal cause a divergence in belief between the two parties, giving rise to informational rents for the agent. Lemma 2 applies without modification, so that the agent benefits from a downward effort deviation after any signal in Y^D, and from upward deviations after any signal in Y^U. The following lemma shows that at least one of these sets is non-empty.

Lemma 5. *Under Assumption 1*, $\ell^k_1 \neq \ell^k_0$ for some $k \in \{1, 2.., K\}$, so that either Y^D or Y^U is non-empty.*

Proof. See Appendix B. □

These two lemmas give us the following theorem.

Theorem 2. *Let Assumptions 1* and 2 hold, assume that the principal is risk-neutral and let* $a^* \in (0,1)$*. For almost all parameters of the model,* $(\mathbf{p}, \mathbf{y})$*, effort* a^* *is* ***not*** *implementable at* $t = 1$*.*

Proof. See Appendix B. □

3.4 Discussion

The fundamental reason for the impossibility result in Theorem 1 is the fact that the agent has continuous as well as discrete choices. In our model, the agent had a continuous choice in the first period (effort) as well as a discrete choice in the second period—stay on the job or quit. The agent's overall value function is given by the maximum of two functions: his payoff in both periods when he stays in period 2, and his overall payoff when he quits. Furthermore, each of these payoff functions is locally linear, in the neighborhood of a^*. The maximum of two linear functions is necessarily convex, and as long as the slopes of the linear functions are unequal, it will also have a kink. Thus, the agent's continuation value as a function of effort has a convex kink. Kinks in the maximum value function arise in other contexts, e.g. when the consumer has discrete choices, but they occur only at an isolated set of prices, and are therefore rare. However, in our agency context, the principal **designs** the contract so as to make the worker indifferent between his discrete choices. Thus convex kink in the maximum value function is inevitable, at precisely the point that is relevant. This phenomenon does not arise in the literature on agency problems with learning, since they assume only discrete actions (Bergemann and Hege (1998, 2005), Hörner and Samuelson (2015), Manso (2011), Kwon (2011) and Bhaskar and Mailath (2019)) or only continuous action where participation constraints do not bind (DeMarzo and Sannikov (2016) and Cisternas (2018)).

Kocherlakota (2004) examines the design of the optimal unemployment insurance with moral hazard and private savings where the principal can fully commit. He assumes that the payoff to the agent is linear in effort and concave in savings. Since the agent can jointly deviate to working less and saving more, the optimal contract must protect against this joint deviation. This implies that the agent's optimization problem is not concave, and thus the first order approach is not valid.[12] In our case, with the absence of commitment, the problem for the principal is significantly worse than the failure of the first-order approach. In a related setting with hidden savings, Chiappori, Macho, Rey and Salanié (1994) show that contracts that imply non-randomized decisions cannot incentivize effort levels that are distinct from the minimum. Park (2004) finds a similar result and complements it with an observation

[12] Arie (2016) examines dynamic moral hazard where effort costs exhibit inter-temporal dependence, and where the principal can commit. He shows that the first order approach fails in this case.

that every equilibrium features the minimum effort if the agent' absolute risk aversion is monotone.

Our analysis does not require any assumptions on the risk preferences of either principal or agent. Thus, the analysis also applies when both parties are risk neutral, so that the principal's cost of providing the worker with utility u is u. Let us informally consider this special case for additional intuition. Suppose that the principal induces first period effort a^*. Since the agent is risk neutral and has no private information on path, it is optimal for the principal to sell the project in the second period at a sum equal to the expected revenue from the project, $R(y, a^*)$, which depends upon beliefs corresponding to the output realization y and first period effort. However, this implies that the agent can increase his second period continuation value by choosing a lower effort $a < a^*$, since there will be some output realization where he is more optimistic regarding the project's profitability than the principal. To deter this, the principal must provide high powered incentives in the first period, so that the agent suffers a first-order loss in flow payoff by choosing $a < a^*$. But in this case, the agent has an incentive to over-work in the first period, since his second period payoff is bounded below by 0, his outside option. This argument assumes that the parties cannot write a long-term contract, where the agent commits to buying the project for both periods. If this is possible, then the problem can be trivially solved, by the agent purchasing the project at price equal to its two-period value, evaluated at the prior belief.

4 Equilibria with random effort

Our results so far have shown that the only deterministic effort levels that the principal can induce in the first period are the extremal ones, 0 and 1, by providing very low or very high powered incentives respectively. Our goal in this section is to show that the principal can, by choosing an intermediate level of incentives, induce random effort with a support that is a non-degenerate compact interval. Moreover, we show that when returns to effort are intermediate, the principal finds it optimal to induce random effort.

Given the complexity of this task, we study a simple case of our model. We assume that both parties, principal and agent are risk neutral. We assume that output is binary and invoke additional assumptions on the information structure and the costs of effort (see Assumption 3 below). We are not aware of a general equilibrium existence theorem that would apply to our model. Consequently, our proof is constructive.

Output takes one of the two values: $y \in \{L, H\}$, with $H > L = 0$, a normalization. Consider a first period contract (u_H, u_L). It is convenient to represent this contract in terms of guaranteed pay u_L and the bonus for high output, $\Omega := u_H - u_L$. At the optimum, the principal sets u_L to make the agent' indifferent between accepting and rejecting the contract. If the agent accepts the contract, his optimal effort is determined by the bonus for high output, Ω. We use the bonus Ω to uniquely identify the contract, since the associated u_L is determined by the agent's participation constraint.

Because output is binary, we omit the superscripts in the notations for probabilities that we used in the previous section. Let $p_{e\omega}$ denote the probability of output H conditional on state ω and effort e. Let ρ measure the difference in the productivity of effort between good and bad states:

$$\rho := (p_{1G} - p_{0G}) - (p_{1B} - p_{0B}).$$

We assume:

Assumption 3.

1. *Effort is more productive in the good state than in the bad state:* $\rho \geq 0$.

2. *Signals satisfy uniform optimism:*

$$\frac{p_{0G}}{p_{0B}} \geq \frac{p_{1G}}{p_{1B}}, \text{ and } \frac{1-p_{0G}}{1-p_{0B}} \geq \frac{1-p_{1G}}{1-p_{1B}}.$$

3. *The function* $c_2''(x)(p_{0G} - p_{0B} + \rho x)$ *is increasing in* x.

We show that a contract with moderate incentives Ω induces a distribution G_Ω over over first period efforts, with support $[\underline{a}_\Omega, \overline{a}_\Omega]$, that is atomless except, possibly, at $\underline{a}_\Omega$. After first period output is realized, the randomness of the first period effort induces two different screening problems, following signals H and L respectively. These screening problems combine moral hazard and adverse selection. The principal's solution to these screening problems give rise to two different continuation values for the agent who chooses first period effort a, $\mathcal{V}_H(a)$ and $\mathcal{V}_L(a)$. These continuation values must be such that $\mathbf{E}[\mathcal{V}_y(a) \mid a]$ plus the first period benefit from a are equal for all values of a in the support $[\underline{a}_\Omega, \overline{a}_\Omega]$. This indifference condition pins down the distribution G_Ω.

Having found a unique continuation equilibrium for every contract Ω, we show that the principal's overall payoff is a continuous function of Ω, so that an optimum exists. We also provide illustrative numerical examples.

Our main result in this section is the following theorem:

Theorem 3. *Suppose Assumption 3 holds. There exists* κ *such that if* $c_1''(a) \geq \kappa$ *for all* $a \in [0,1]$*, any first-period contract* Ω *gives rise to a unique continuation equilibrium. The agent optimally chooses first period effort as follows:*

- *Effort* 0 *if* $\Omega \leq \underline{\Omega} := c'(0)$
- *Effort* 1 *if* $\Omega \geq \overline{\Omega} > c'(1)$

- *A random effort with support* $[\underline{a}_\Omega, \bar{a}_\Omega]$ *and distribution* G_Ω *if* $\Omega \in (\underline{\Omega}, \overline{\Omega})$.

There exists a profit maximizing first-period contract, Ω^**, for the principal.*

In Sections 4.1-4.3, we provide a constructive proof for Theorem 3. Along the way we explain the intuition and describe the details of the equilibrium: we characterize the distribution G_Ω and the equilibrium second period screening contracts.

Let us briefly discuss the role of Assumption 3 deriving our results. The first part of the assumption, namely $\rho \geq 0$, allows us to show that local incentive compatibility[13] ensures global incentive compatibility in the second period screening problem, as long as the second period bonus is increasing in the agent's private belief. This is shown in Lemma 6. The uniform optimism assumption is stronger than Assumption 1 in the previous section, and ensures that when the agent randomizes first period effort, the "type" who chooses the highest effort in the support is most pessimistic about the project after both signals.[14] Consequently, this type earns zero rents in the second period, allowing us to pin down the largest effort in the support of the distribution—it must maximize the agent's first period payoff given the first period bonus Ω. The third part of Assumption 3 ensures that the principal's objective in the second period problem is concave. Note that informational assumptions (parts 1 and 2 in Assumption 3) are satisfied if technology is equally productive in both states, so that $\rho = 0$. This is the linear-additive structure that is similar to Holmström (1999) or Milgrom and Roberts (1992).

One difficulty is worth mentioning: we are not aware of any general equilibrium existence results for the subgame game following the principal's choice of first period bonus, Ω. If we were assured of equilibrium existence, and if this needed no further assumptions, then the solution to the differential equation that arises from the agent's indifference condition—that he should get the same payoff from any effort in the support of G_Ω—would be an equilibrium. We would not need to worry about global incentive compatibility, and the first and third parts of Assumption 3 would not be needed. However, in the absence of a general existence result we do need to show that the unique solution to the differential equation is not vulnerable to large deviations by the agent in the second period.

4.1 The final period screening menu

In this section we do two things:

1. We set up the optimality conditions for the second-period contracts. The principal uses a screening menu of contracts to elicit the agent's private beliefs about the difficulty of the task.

[13] That the agent's value function satisfies the envelope condition.

[14] Under Assumption 1, Lemma 3 ensures that highest effort type is the most pessimistic after at least one of the two signals.

2. We derive the relationship between the first-period effort, the observed signal and the agent's private belief. This relationship is needed to characterize the agent's continuation value as a function of effort a.[15]

At the end of period one, both parties observe the realization of the output signal, $y \in \{L, H\}$. We condition the analysis of this subsection on this realization without, for notational simplicity, making it explicit.

Recall that the effort is private, and therefore, the agent's posterior belief following the first-period signal is private too. From the principal's perspective, the agent's belief μ is distributed on the interval $\mathcal{I} = [\underline{\mu}, \overline{\mu}]$, according to some distribution $F(\mu)$.

The principal responds to the agent's private information by setting up a menu $\big(u_L(\mu), \Delta(\mu)\big)_{\mu \in \mathcal{I}}$, where $\Delta(\mu) = u_H(\mu) - u_L(\mu)$ is the bonus that incentivises the agent to exert effort. An item in the menu is indexed by a belief-type μ that is supposed to choose it. Incentive compatibility trivially implies that if $\Delta(\pi) > \Delta(\mu)$, then $u_L(\pi) < u_L(\mu)$, so that we may identify a contract in the menu by its bonus $\Delta(\mu)$ alone.

Suppose that belief-type μ accepts some contract Δ (which may or may not be optimal for this belief-type μ). Let $b(\mu, \Delta)$ denote optimal effort for this belief type under Δ. If this is neither zero nor one, it satisfies the first order condition[16]

$$M(\mu)\Delta = c_2'\big(b(\mu, \Delta)\big), \tag{3}$$

where $M(\mu)$ denote the rate at which effort increases the probability of high output at belief μ:

$$M(\mu) := (p_{1B} - p_{0B}) + \mu\rho.$$

Naturally, optimal effort is zero if the incentives are low, i.e., if $M(\mu) < c_2'\big(0\big)$. Optimal effort is one if the incentives are large: $M(\mu) > c_2'\big(1\big)$.

Note that $b(\mu, \Delta(\mu))$ is the effort of the belief-type μ when he chooses an incentive-compatible contract meant for him.[17] To set up the principal's maximization problem we characterize the set of incentive compatible menus—i.e., menus for which a belief-type μ selects a contract $\Delta(\mu)$. Let $V(\mu, \hat{\mu})$ denote the (second-period) payoff of the agent under an incentive compatible menu when agent's belief is μ, but he chooses a contract that is designed for an agent with belief $\hat{\mu}$. By the envelope theorem,

$$\frac{d}{d\mu}V(\mu, \mu) = \Delta(\mu)\big[(p_{0G} - p_{0B}) + b(\mu, \Delta(\mu))\rho\big]. \tag{4}$$

Lemma 6. *Suppose that $\rho \geq 0$. A menu $\big(u_L(\mu), \Delta(\mu)\big)_{\mu \in \mathcal{I}}$ is incentive compatible if $\Delta(\mu)$ is increasing and the envelope condition on the associated value (4) holds.*

[15] The continuation value of first period effort a equals the expected second-period information rent awarded to type a by the pair of screening menus, one after each output realization.

[16] Since the rewards are linear in effort while the costs of effort are convex, the second order conditions are satisfied.

[17] The recommended effort, in Myerson's language.

Proof. See appendix C. □

Remark 3. *Chade and Swinkels (2021) analyze the interplay of moral hazard and adverse selection more generally, and provide conditions under which it suffices to consider deviations where type μ chooses the recommended effort for type $\hat{\mu}$ when choosing contract $\Delta(\hat{\mu})$. Our approach is more direct, since we allow μ to choose his optimal effort for contract $\Delta(\hat{\mu})$.*

The principal designs the menu that maximizes her second-period expected payoff

$$\max_{\Delta(\cdot)\in\mathcal{D}} \int_{\underline{\mu}}^{\overline{\mu}} \left[\mathbf{p}_{b(m,\Delta(m)),m} \cdot \mathbf{y} - c_2(b(m,\Delta(m))) - \frac{1-F(m)}{f(m)}\frac{d}{dm}V(m,m)\right] dF(m), \quad (5)$$

where $\mathcal{D}$ is the set of all incentive compatible contracts, i.e., contracts $\Delta(\mu)$ that are increasing functions of μ. The expression under the integral is a virtual surplus: it is a surplus generated by contract $\Delta(m)$ net of term that reflects the cost of making the menu incentive compatible. This cost arises because the agent with private information earns an informational rent.

The principal maximizes her second period payoff in (5) point-wise. Under Assumption 3, at each belief m, the objective is a concave function of b, and therefore, the first order condition is both necessary and sufficient for b (and the corresponding Δ) to be a solution. Of course, our method of constructing an equilibrium is valid only if the solution to the point-wise maximization of (5) satisfies the monotonicity requirement, namely that $\Delta(\mu)$ is increasing. We verify that this requirement holds in Lemma 9.

4.1.1 From beliefs to actions

Our next step is to translate our findings on the agent's beliefs to his first period actions, since the first period indifference condition for the agent pertains to actions. Given a first period output realization $y \in \{L, H\}$, Bayes rule yields a one-to-one map between the agent's first period action a and his belief, $\mu_y(a)$:

$$\mu_H(a) = \frac{\lambda(p_{1G}a + p_{0G}(1-a))}{\lambda(p_{1G}a + p_{0G}(1-a)) + (1-\lambda)(p_{1B}a + p_{0B}(1-a))},$$
$$\mu_L(a) = \frac{\lambda[(1-p_{1G})a + (1-p_{0G})(1-a)]}{\lambda[(1-p_{1G})a + (1-p_{0G})(1-a)] + (1-\lambda)[(1-p_{1B})a + (1-p_{0B})(1-a)]}.$$

We now rewrite the principal's second period problem in terms of first period effort rather than agent's beliefs. Compared to other models of screening with endogenous types, public signals make the problem rather complicated: there are two different screening menus following signals H and L, and the optimality conditions for the menus are tangled together via the endogenous distribution of the first-period effort. This implies that the first period indifference condition that pins down this distribution is a functional equation.

To solve it, we rely on a property of **regular** screening problems—i.e., problems for which the distribution of agent's belief-types induces a strictly monotone menu of contracts. For regular screening problems, as far as the distribution of types is a concern, the optimal contract depends only on the inverse reverse hazard rate of this distribution. We use this property to convert the previously mentioned functional equation into an ordinary differential equation for the inverse reverse hazard rate. We validate this approach by providing conditions on the parameters of the model under which the inverse reverse hazard rate induces a regular screening problem.[18]

Let $G_\Omega(a)$ be the distribution of the first period effort under the contract Ω. We assume that it has support $[\underline{a}_\Omega, \overline{a}_\Omega]$ and that it is continuous everywhere, except, perhaps, at $\underline{a}_\Omega$, and that it admits a density $g_\Omega(a)$ for the continuous part of the distribution. We later verify that all these assumptions hold: this distribution arises as a solution to a differential equation that represents the agent's indifference condition.

Under the uniform optimism, beliefs are decreasing in the action, and the distributions of the first period effort and second period beliefs following signal y are connected via the following equations:

$$F_{\Omega,y}\big(\mu_y(a)\big) = 1 - G_\Omega(a),$$
$$f_{\Omega,y}\big(\mu_y(a)\big) = \frac{g_\Omega(a)}{-\mu_y'(a)}.$$

Let $M_y(a) = M(\mu_y(a))$.

The inverse reverse hazard rate for the distribution G_Ω is

$$h_\Omega(a) = \frac{G_\Omega(a)}{g_\Omega(a)}.$$

We replace the distribution of beliefs with the inverse reverse hazard rate $h_\Omega(a)$, solve for it, and then find the corresponding distribution G_Ω.

Let us rewrite the virtual surplus—the expression under the integral in (5)—as a function of the first period effort and replace the inverse reverse hazard rate with a parameter h and define $\tilde{b}_y(a,h)$ to be the value of b that maximizes it:[19]

$$\tilde{b}_y(a,h) = \arg\max_{b\in[0,1]} \left\{ \mathbf{p}_{b\mu_y(a)} \cdot \mathbf{y} - c_2(b) + h\mu_y'(a)(\rho b + p_{0G} - p_{0B}) \frac{c'(b)}{M_y(a)} \right\}.$$

When $h = h_\Omega(a)$, $\tilde{b}_y(a,h)$ is the effort induced by the optimal menu. However, we extend it to be defined for arbitrary inverse reverse hazard rates to capture what happens to the induced effort if we perturb the distribution G_Ω.

[18]See Lemmas 8 and 9.

[19]Note that if a menu maximizes virtual surplus for every belief μ, it maximizes the principal's objective.

Because the objective is concave, $\tilde{b}$ is an interior maximizer for a given a and h if and only if it solves

$$y[M_y(a)]^2 = c_2'(\tilde{b})M_y(a) - \mu_y'(a)h\Big[(c_2'(\tilde{b}) + \tilde{b}c_2''(\tilde{b}))\rho + c_2''(\tilde{b})(p_{0G} - p_{0B})\Big]. \quad (6)$$

There exists a unique differentiable solution to this equation because the right-hand side is continuous and strictly increasing in $\tilde{b}$. Let $\beta_y(a,h)$ be the solution. If $\beta_y(a,h)$ is negative for some values of a and h, then it means that the incentives to exert effort are insufficient and the optimal effort for those values is zero. To account for it, let

$$\tilde{b}_y(a,h) = \max\{0, \beta_y(a,h)\}.$$

Given a distribution of the first period effort, this formula characterizes the optimal screening menu through the effort it induces. The contracts that form the optimal menu can be found using equation (3).

We define a threshold for the inverse reverse hazard rate $\tilde{h}_y(a)$ such that if the inverse reverse hazard rate for a given a is above this threshold, the effort induced following the signal y is zero:

$$\tilde{h}_y(a) := \frac{H[M_y(a)]^2 - c_2'(0)M_y(a)}{-\mu_y'(a)\Big[c_2'(0)\rho + c_2''(0)(p_{0G} - p_{0B})\Big]}. \quad (7)$$

The threshold $\tilde{h}_y(a)$ plays an important role: it determines the conditions under which the agent is not incentivized by the principal—this occurs when the agent is sufficiently pessimistic about the state of the world. Later, when we solve the differential equation for the inverse reverse hazard rate, we use these thresholds as a boundary condition, because the agent with the most pessimistic belief is not incentivized to exert effort in the second period.

The effort that maximizes principal's payoff is decreasing in the inverse reverse hazard rate h and the past effort a. This is intuitive and immediate. A larger h implies a greater cost in terms of informational rents to lower effort types. Similarly, a larger a implies more pessimistic beliefs, and a lower return to effort given that $\rho \geq 0$. This is formalized in the lemma below.

Lemma 7. *Under Assumption 3, function $\tilde{b}_y(a,h)$ is decreasing in both arguments: $\frac{\partial \tilde{b}_y(a,h)}{\partial a} \leq 0$ and $\frac{\partial \tilde{b}_y(a,h)}{\partial h} \leq 0$.*

Proof. See Appendix C. □

The required monotonicity condition for global incentive compatibility in (6) pertains to Δ, namely that it should be increasing in the agent's belief μ, or equivalently, decreasing in his first period action a. The previous lemma implies that the effort induced by the principal is increasing in the agent's belief. Of course, the effort induced and Δ are linked, since the agent equates his marginal cost effort to the marginal return, as in equation (3). The following lemma establishes that global incentive compatibility will be satisfied as long as the inverse reverse hazard rate is increasing. Its proof invokes part 3 of Assumption 3.

Lemma 8. *If $h_\Omega(a)$ is increasing in a and Assumption 3 holds, then $\Delta_{\Omega,H}(\mu)$ and $\Delta_{\Omega,L}(\mu)$ are both increasing in μ, satisfying global incentive compatibility.*

Proof. See Appendix C. □

Let us sum up the results of this section so far: we have constructed the optimal second period screening menu as a function of the inverse reverse hazard rate of the first period effort. We have also shown that a sufficient condition for global incentive compatibility, in each of the second period screening problems, is that the inverse reverse hazard rate of first period effort is increasing.

Our next step is to derive conditions that pin down the distribution of first period effort, as a function of first period incentives Ω. The agent's second period payoff after any first period effort depends upon the distribution or inverse reverse hazard rate of his first period effort. We look for the distribution with the following property: when the agent increases his first period effort, the resulting increase in his first period payoff must be exactly offset by a reduction in the expected information rent in the second period.

4.2 The first period effort

Given incentive pay Ω, the agent must be indifferent between all efforts in the interval $[\underline{a}_\Omega, \overline{a}_\Omega]$. When the agent chooses the highest equilibrium effort level $\overline{a}_\Omega$, he is the most pessimistic type after both signal realizations, due to our assumption of uniform optimism, and his second period continuation value is zero. In the second period, this type is excluded following any realization of the signal: he chooses a contract $\Delta_{\Omega,y}(\mu_y(\overline{a}_\Omega)) = 0$ and exerts no effort.

Thus $\overline{a}_\Omega$ must be myopically optimal given Ω, pinning down $\overline{a}_\Omega$:

$$\Omega M(\lambda) = c_1'(\overline{a}_\Omega).$$

The condition that every effort level in $[\underline{a}_\Omega, \overline{a}_\Omega]$ yields the same overall payoff is:

$$\Omega M(\lambda)(\overline{a}_\Omega - a) - [c_1(\overline{a}_\Omega) - c_1(a)] = \Pr(H|a)\mathcal{V}_H(a) + \Pr(L|a)\mathcal{V}_L(a), \qquad (8)$$

where $\mathcal{V}_y(a) := V_y(\mu_y(a), \mu_y(a)),\ y \in \{L, H\}$ is the second period payoff of the agent who exerted effort a and produced output y in period 1.

The value functions on the right-hand side of the above equation depend on the distribution of effort through the inverse reverse hazard rate $h_\Omega(a)$. We use the fact that $\mathcal{V}_y(\overline{a}_\Omega) = 0$ and equation (4) to obtain

$$\mathcal{V}_y(a) = \int_a^{\overline{a}_\Omega} \frac{-\mu_y'(z)c_2'(\tilde{b}_y(z, h_\Omega(z)))}{M_y(z)}[p_{0G} - p_{0B} + \rho\tilde{b}_y(z, h_\Omega(z))]dz.$$

Also, recall that

$$\Pr(H \mid a) = \lambda(ap_{1G} + (1-a)p_{0G}) + (1-\lambda)(ap_{1B} + (1-a)p_{0B}).$$

By plugging these expressions in (8), we arrive at the equation for the inverse reverse hazard rate $h_\Omega(a)$. The following lemma shows that the solution exists and fulfills the monotonicity requirement that makes the screening problems regular (see Lemma 8). In addition, the proof for Lemma 9 offers a practical recipe for finding an equilibrium, namely, a way to rewrite agent's indifference condition (8) as an ordinary differential equation for inverse reverse hazard rate.

Lemma 9. *Equation* (8) *has a unique solution* $h_\Omega(a)$. *The solution is jointly continuous in* (a, Ω). *Moreover, there exists* κ *such that if* $c_1''(a) \geq \kappa$ *for all* $a \in [0,1]$, *then* $h_\Omega(a)$ *is increasing in* a.

Proof. See Appendix C. □

Given the solution h_Ω of equation (8), we can find the distribution of the first period efforts. Depending on the magnitude of Ω, there are two cases. If $h_\Omega(a)$ is discontinuous at $\underline{a}_\Omega$ (which is the case when Ω is small and $\underline{a}_\Omega = 0$), then the distribution of efforts has a mass point at 0 and continuous everywhere else. Otherwise the distribution of efforts is continuous. In either case:

$$G_\Omega(a) = \begin{cases} 0, & \text{if } a < \underline{a}_\Omega, \\ \lim\limits_{a \to \underline{a}_\Omega + 0} e^{-\int_a^{\bar{a}_\Omega} \frac{1}{h_\Omega(x)} dx}, & \text{if } a = \underline{a}_\Omega, \\ e^{-\int_a^{\bar{a}_\Omega} \frac{1}{h_\Omega(x)} dx}, & \text{if } a \in (\underline{a}_\Omega, \bar{a}_\Omega), \\ 1 & \text{if }, a \geq \bar{a}_\Omega. \end{cases}$$

This concludes our construction of the equlibrium **given** some first period contract Ω. Let us discuss how the the problem of the convex kink, that arose with deterministic effort, is solved. The key idea is that the distribution G_Ω controls informational rents. The largest equilibrium effort, $\bar{a}_\Omega$, maximizes the agent's first period payoff, and thus upward deviations are unprofitable. The subtlety arises in ensuring that the left-hand derivative of the agent's expected continuation value is zero at $\bar{a}_\Omega$, which can be achieved if the principal induces zero effort from type $\bar{a}_\Omega$ after both signals, and sets the bonus to zero for this type of agent. This requires that the density at and near $\bar{a}_\Omega$ is small enough. If the lowest equilibrium effort, $\underline{a}_\Omega = 0$, downward deviations are moot. If not, at $\underline{a}_\Omega$, the loss in expected informational rent from marginal additional effort exactly equals the gain in current payoff given the bonus. Consequently, the expected continuation value function is now differentiable everywhere, including at the boundaries $\underline{a}_\Omega$ and $\bar{a}_\Omega$, thereby overcoming the impossibility that arose with deterministic effort.

Our analysis shows that the set of implementable mixed effort profiles is severely circumscribed. The support of any mixed effort profile is a compact interval. The upper bound, $\bar{a}_\Omega$, is uniquely defined by the bonus, Ω, and is the maximizer of the agent's period 1 payoff. For sufficiently small Ω, the lower bound is 0, in which case there may be a mass point at 0—this is the only possible mass point. Otherwise, if Ω is larger, then the lowest effort, $\underline{a}_\Omega > 0$, and the distribution G_Ω is atomless. The distribution G_Ω necessarily induces regular screening problems after any signal: apart from excluding agents with particularly pessimistic beliefs, the contracts are strictly monotone.

4.3 The optimal first period contract

If $\Omega \leq \underline{\Omega} := c_1'(0)$, the agent has no incentive to exert effort and chooses $a = 0$ for sure. Any Ω satisfying this inequality yields the same payoff to the principal, since both parties are risk neutral and the principal can choose u_L to hold the agent to his reservation utility. Similarly, there exists $\overline{\Omega} > c_1'(1)$ that any $\Omega > \overline{\Omega}$ induces $a = 1$ with certainty.[20] Once again, since the principal chooses u_L to hold the agent to his reservation utility, and since the parties are risk-neutral, any $\Omega > \overline{\Omega}$ yields the same profit. We may therefore limit search of the principal's optimal contract to the compact interval $[\underline{\Omega}, \overline{\Omega}]$.

Let $B_y(a, \Omega) := \tilde{b}_y(a, h_\Omega(a))$. The principal's total payoff is the sum of his payoffs over the two periods. The second-period payoff is the maximized value of the principal across the two screening problems that we solved in the previous section:

$$\pi_{2y}(a, \Omega) = -\mu_y'(a) \left[\mathbf{p}_{B_y(a,\Omega),\mu_y(a)} \cdot \mathbf{y} - c_2(B_y(a, \Omega)) + \mu_y'(a) h_\Omega(a) \frac{d}{d\mu} V_y(\mu, \mu) \Big|_{\mu=\mu_y(a)} \right],$$

and

$$\pi_2(a, \Omega) = \Pr(H|a)\pi_{2H}(a, \Omega) + \Pr(L|a)\pi_{2L}(a, \Omega).$$

The first period joint surplus, as a function of the agent's effort, is:

$$\pi_1(a) = \mathbf{p}_{a\lambda} \cdot \mathbf{y} - c_1(a).$$

First period effort has distribution G_Ω, and this may have a mass point at its lower bound, $\underline{a}_\Omega$. Therefore, the total expected payoff of the principal is

$$\pi(\Omega) = G_\Omega(\underline{a}_\Omega)[\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega, \Omega)] + \int_{\underline{a}_\Omega}^{\bar{a}_\Omega} [\pi_1(a) + \pi_2(a, \Omega)] g_\Omega(a) da.$$

Note that every item on the right-hand side of the equation depends on Ω.

[20] $\overline{\Omega} = c_1'(1) - W_1^-(1, 1)$, since the contract must prevent the agent from shirking in order to raise his second period continuation value (see Section 3.2).

Lemma 10. *$\pi(\Omega)$ is continuous on $[\underline{\Omega}, \overline{\Omega}]$, and therefore, there exists an optimal first period contract, i.e., contract $\Omega^* \in [\underline{\Omega}, \overline{\Omega}]$ that maximizes $\pi(\Omega)$.*

Proof. See Appendix C. □

The argument in Lemma 10 completes the proof of the main theorem in this section. In summary, the equilibrium strategies are as follows:

1. For every Ω, the agent chooses first period effort a according to $G_\Omega(a)$.
2. For each output realization y, the principal offers a second period menu of contracts $\Delta(m)$ for each private belief m:

$$\Delta_{\Omega,y}(m) = \frac{c_2'\big(\tilde{b}_y(\mu_y^{-1}(m), h_\Omega(\mu_y^{-1}(m)))\big)}{M_y(m)}.$$

3. If the agent has chosen first period effort in the support of G_Ω, he chooses the contract designed for his belief type. [21]
4. The agent chooses second period effort $b(\mu_y(a), \Delta)$.
5. Finally, the principal chooses the first period contract Ω^* that maximizes $\pi(\Omega)$.

We conclude with an observation that is relevant for the theory of organizational forms. Since the agent is risk neutral, one might naively suppose that the principal could sell the firm to the agent in the second period. However, this would severely circumscribe the principal's options in the dynamic context. If the principal were to sell the firm, she would have to post a price, and cannot control the agent's effort decision after purchase. This severely restricts her ability to screen the worker's first period effort.

4.4 The inefficiency of random effort

First-best effort in our model can be found by assuming that the agent owns the project.[22] Second-period effort maximizes returns net of the effort cost in all cases. If $\rho = 0$, second period effort will be independent of beliefs, since effort is equally productive in both states. Consequently, there is no learning benefit in the first period, and first period effort must also maximize the net expected return, and will the same as second period effort. Thus learning is entirely detrimental when $\rho = 0$, due to the ratchet effect. If $\rho > 0$, higher effort is more informative about project

[21] If the agent has chosen outside the support of G_Ω, his optimal choice is to either reject the menu (if a is greater than the upper bound) or to choose the contract designed for the highest on-path belief type.

[22] Alternatively, we can assume that the principal retains control, but that the agent's first-period effort is observable but not contractible.

quality than low effort, and this raises first best first period effort upwards, above the static optimum.

Quite different considerations are at play in the equilibrium contract. Note that the principal does not care per se about the agent's informational rents in the second period, since she is able to collect these rents upfront in the first period. However, effort distortions in the second period do affect her payoff (as do first period effort distortions). Consequently, the principal has a bias towards extremal efforts, which ensure that the agent has no private information in the second period, thereby allowing her to implement first-best effort in the second period. In other words, first period effort can be distorted upwards or downwards, towards extremal efforts. Even when the principal induces a distribution over efforts, there is a benefit to the principal from moderate first period bonus. For in this case, the consequent distribution G_Ω has a mass point at zero effort, and the second-period effort conditional on choosing zero will be efficient. More generally, if the principal induces random effort, we note that the first period effort is necessarily inefficient, since the first-best effort is unique. The second-period effort is also distorted downwards for every type $a > \underline{a}_\Omega$.

We now compare our results with those in the literature on the ratchet effect where the worker has private information at the outset, i.e. Laffont and Tirole (1988). Their main result is that there will be substantial pooling of types in the first period, and thus effort will be distorted downwards in the second period, since the agent retains private information. Similarly, in our setting, when the principal optimally induces mixed effort, second period output is distorted downwards. In the first period, Laffont and Tirole (1988) find that due to pooling, some types may asked to produce too much, while others are asked to produce too little. In our setting, there is but a single type of agent in the first period; however, since the principal has a bias towards inducing extreme efforts, as a way of minimizing the inefficiencies arising from agent private information in the second period, one may have excessive effort or insufficient effort.

4.5 Illustrative example

We now provide an illustrative numerical example. Assume quadratic effort costs, $c_1(e) = c_2(e) = \frac{e^2}{2}$, and set the other parameters of the model to

$$\begin{aligned} &p_{1G} = 0.9, \quad p_{0G} = 0.6, \\ &p_{1B} = 0.2, \quad p_{0B} = 0.1, \\ &\lambda = 0.5, \qquad H = 2,\ L = 0. \end{aligned}$$

Suppose that the first period contract has $\Omega = 1$. This contract is chosen for illustration; it provides intermediate incentives—i.e., $\underline{\Omega} < \Omega < \overline{\Omega}$—but otherwise, is arbitrary.

We use equation (7) to identify when the principal induces an interior effort in the second period. Namely, if the inverse reverse hazard rate is below $\tilde{h}_y(a)$, the effort following the signal y should be strictly positive; otherwise, it is zero:

$$\tilde{b}(a,h) = 0 \iff h \geq \tilde{h}_y(a) = \frac{y_h[M_y(a)]^2}{-\mu_y'(a)(p_{0G} - p_{0B})}.$$

This inequality defines the three regions for the inverse reverse hazard rate for which the induced effort is (see Figure 1):

(a) zero after both H and L signals;

(b) zero only following signal L;

(c) positive following both H and L signals.

Recall that $\tilde{h}_y$ defines a boundary condition for equation (8) which we solve using the quadrature method. At the upper bound of the support $\overline{a}_\Omega$, the induced effort is zero following both signals, therefore $h_\Omega(\overline{a}_\Omega) = \tilde{h}_H(\overline{a}_\Omega)$.

Given the parameters of the model and contract $\Omega = 1$, the support of the effort distribution is $[0.0727, 0.2]$. The resulting inverse reverse hazard rate $h_\Omega(a)$, distribution of first period efforts $G_\Omega(a)$, induced second-period efforts $\tilde{b}_H\left(a, h_\Omega(a)\right)$ and $\tilde{b}_L\left(a, h_\Omega(a)\right)$, and the second period incentives $\Delta_{\Omega,H}\left(\mu_H(a)\right)$ and $\Delta_{\Omega,L}\left(\mu_L(a)\right)$ are depicted on Figure 1.

Note that the support of the first period effort is partitioned into two regions:

1. for sufficiently high effort a, the principal induces positive second period effort only when signal H is realized;

2. for low effort a, the principal provides enough incentives for effort to be positive regardless of the signal, but the effort induced after signal H is the higher of the two.

This pattern of efforts can be explained using two important features of the model: on the one hand, the agent should receive rent in the second period that decreases to zero when her first period effort approaches $\overline{a}_\Omega$ from below. This ensures that $\bar{a}_\Omega$ maximizes the agent's first period payoff, and that his continuation value function is differentiable at this point. On the other hand, effort is more productive after signal H. Therefore, if for some values of a, induced effort $\tilde{b}_H$ is close to zero, the induced effort $\tilde{b}_L$ must be zero, since the beliefs are more pessimistic after L, and the the principal's return to effort is lower.

Next, we illustrate how the principal's overall payoff depends on the first period incentives. As shown on Figure 1, the principals payoff is maximized at the intermediate level of the first period incentives $\Omega^* = 2.53$. When the optimal contract is offered in equilibrium, the first period effort is distributed on $[0.38, 0.52]$. This is the case when the principal finds it optimal to induce random efforts in the first period: on the one hand, the difference in rewards between outcomes H and L is not sufficiently

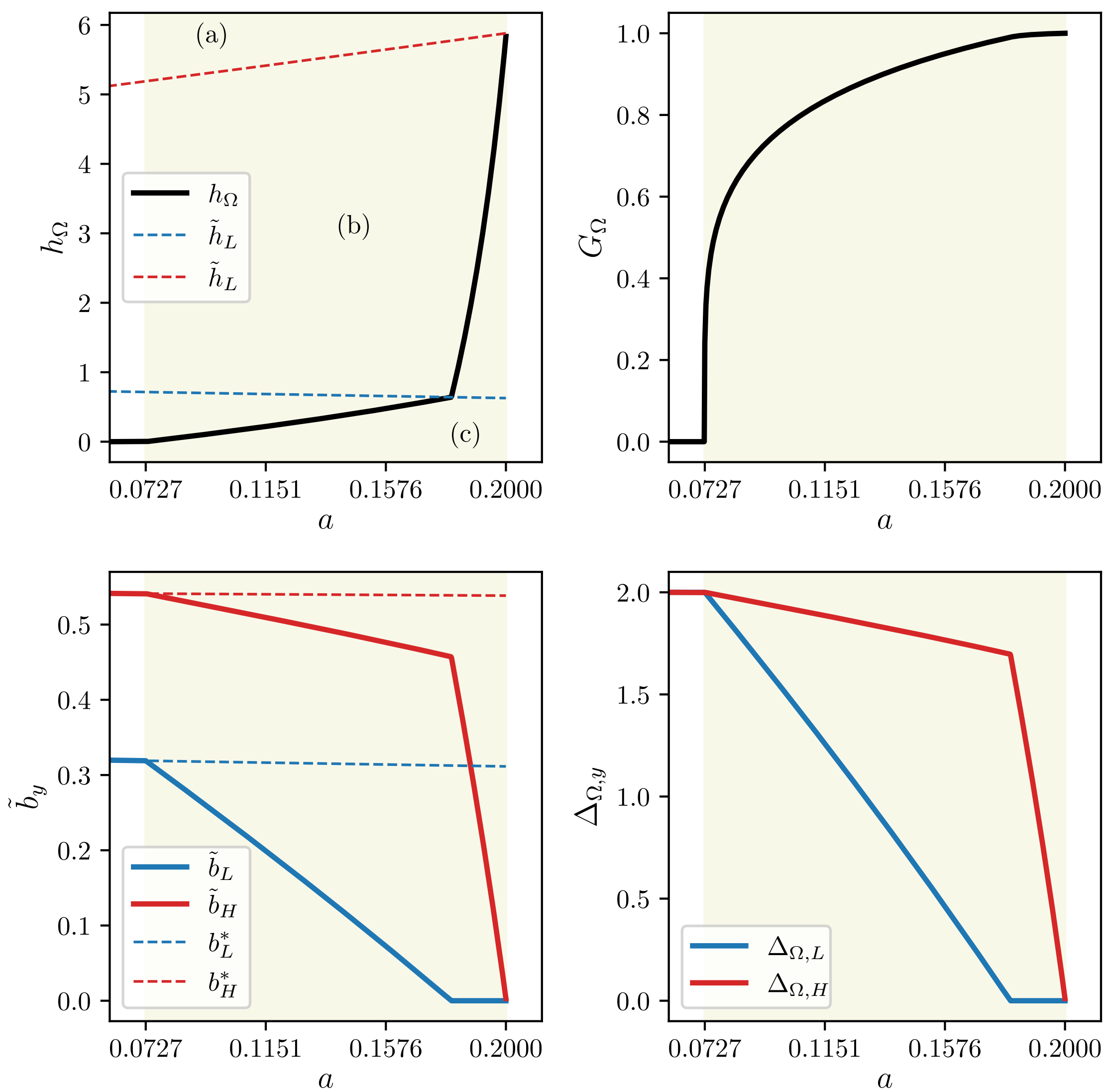

Figure 1: Equilibrium second-period contract and agent's effort choice conditional on the first period contract $\Omega = 1$.

Note: Top left is the inverse reverse hazard rate h_Ω (dashed lines indicated the borders of the three regions; see equation (17)); Top right is the distribution of first period efforts G_Ω; Bottom left is the induced second-period efforts $\tilde{b}_H$ and $\tilde{b}_L$ (dashed lines indicate conditional first-best effort levels); Bottom right indicate second-period incentives $\Delta_{\Omega,H}$ and $\Delta_{\Omega,L}$; shaded region indicates the support of G_Ω; blue lines indicate choices after output L and red lines indicate choices after output H.

large for the principal to induce effort 1, and on the other hand, the marginal cost of effort is zero at zero, so it is never optimal to let the agent shirk. Note that the first-best level of effort $a^{fb} = 0.43$ lies in the interior of the support.[23]

Consider a modified example, in which the gain from a successful project is much higher at $H = 3.65$, and the rest of the parameters are the same. For this parametrisation, the first-best level of effort is $a^{fb} = 0.82$, but the principal finds it optimal to induce extremal effort $a = 1$ in the first period. As shown on Figure 1, the principal achieves this with a contract $\Omega^* = 6.6 > 1 = c'(1)$ (or higher). The difference between the contract incentive and the marginal cost of effort prevents the agent from secretly reducing his effort in the first period and capitalizing on his relative optimism later. In the second period, the principal induces the conditional first-best efforts by offering the contract $\Delta_H = \Delta_L = H$ and extracts the full rent from the agent.

Note that in both cases, the optimal incentives $\Omega^* > H$. Although, it is true in general, the intuition for this is especially clear in the second case. If the principal were to offer $\Omega = H$, the agent would exploit the contract by shirking in the first period and, as a result, would become more optimistic than the principal when the public signal is realized. Since the second period contract is determined by the principal's equilibrium beliefs about the agent's productivity, the agent would benefit from exerting an effort that is higher than expected. To make sure that such behavior is not optimal, the principal must extra high-powered incentives in period 1.

The difference between the first and the second cases of this example is subtle: in the second case, the principal ensures that such a behavior does not emerge; in the first case, the principal ensures that such a behavior does not emerge more often than indicated by the equilibrium distribution $G_{\Omega^*}(a)$. In the latter case the principal allows for some degree of asymmetric information. However, in contrast with our impossibility result in Theorems 1 and 2, the asymmetry of information is anticipated and correctly quantified as the principal expects the agent to exert effort below $\overline{a}_{\Omega^*}$ in order to collect rent in final period.

5 Conclusion

We have studied the ratchet effect where principal and agent are both uncertain about the technology. No interior deterministic effort can be implemented in the first period, and consequently, the optimal contract involves extremal efforts or random effort. Thus effort is distorted, and this may deter the introduction of new technology. Our negative results, on the non-implementability of deterministic effort, are quite general. However, the analysis of random effort is quite complex since we have the interaction of moral hazard and adverse selection, where the agent's private information is generated both by private effort and public output. This combination

[23]The first-best level of effort is different from the myopic level of effort that solves $HM(\lambda) = c'(a)$ because of learning.

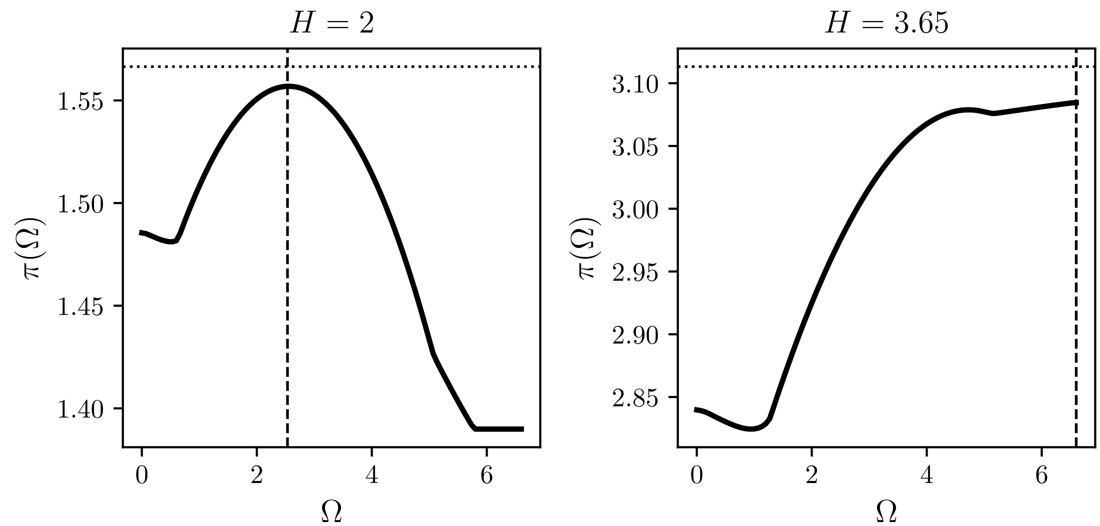

Figure 2: The principal's payoff conditional on the first period contract Ω.

Note: Left is the payoff for the case of $H = 2$; Right is the payoff for the case of $H = 3.65$. Dotted line is the principal's first-best payoff.

leads to complications that do not arise in other contexts such as the hold-up problem with unobserved investments. Thus, we have to invoke stronger assumptions in our analysis of random effort, and relaxing these would be an important future task.

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Appendix A: Proofs related to Theorem 1

Proof of Lemma 1

The optimal contract in the final period $\mathbf{u}$ must satisfy the first order conditions for $\hat{b}$ to be optimal at μ: [24]

$$\mathbf{u} \cdot (\mathbf{p}_{1\mu} - \mathbf{p}_{0\mu}) = c_2'(\hat{b}) > 0. \tag{9}$$

In an optimal static contract, utility payments u^k must be increasing, since they are ordered in terms of the likelihood ratio. Thus $\mathbf{u}$ can be written as $\mathbf{u} = z \cdot \mathbf{1} + \tilde{\mathbf{u}}$, where $z \cdot \mathbf{1}$ is a vector where each component equals z, and $\tilde{u}^k > 0$ if $y^k \in Y^H$, and $\tilde{u}^k < 0$ if $y^k \in Y^L$.The agent's payoff from his optimal effort choice at π is no less than his payoff from choosing $\hat{b}$ at π, which equals

$$\left[\mathbf{u} \cdot \mathbf{p}_{\hat{b}\pi} - c_2(\hat{b})\right] - \left[\mathbf{u} \cdot \mathbf{p}_{\hat{b}\mu} - c_2(\hat{b})\right] = (\pi - \mu)\mathbf{u} \cdot (\mathbf{p}_{\hat{b}G} - \mathbf{p}_{\hat{b}B}) = (\pi - \mu)\tilde{\mathbf{u}} \cdot (\mathbf{p}_{\hat{b}G} - \mathbf{p}_{\hat{b}B}),$$

since $\mathbf{1} \cdot (\mathbf{p}_{\hat{b}G} - \mathbf{p}_{\hat{b}B}) = 0$. Assumption 1 implies that $p^k_{\hat{b}G} - p^k_{\hat{b}B} > 0$ if $y^k \in Y^H$ and $p^k_{\hat{b}G} - p^k_{\hat{b}B} < 0$ if $y^k \in Y^L$, and thus $\tilde{\mathbf{u}} \cdot (\mathbf{p}_{\hat{b}G} - \mathbf{p}_{\hat{b}B}) > 0$.

Letting $\tilde{b}(\pi)$ denotes the optimal effort choice at belief π,

$$\hat{V}(\pi, \mu) = \mathbf{p}_{\tilde{b}(\pi)\pi} \cdot \mathbf{u} - c_2(\tilde{b}(\pi)).$$

The derivative with respect to π, evaluated at (π, μ), equals

$$\begin{aligned} V_1(\pi, \mu) &= \left(\mathbf{p}_{\tilde{b}(\pi)G} - \mathbf{p}_{\tilde{b}(\pi)B}\right) \cdot \mathbf{u} + \frac{d\tilde{b}}{d\pi}\left[(\mathbf{p}_{1\pi} - \mathbf{p}_{0\pi}) \cdot \mathbf{u} - c_2'(\tilde{b}(\pi))\right] \\ &= \left(\mathbf{p}_{\tilde{b}(\pi)G} - \mathbf{p}_{\tilde{b}(\pi)B}\right) \cdot \mathbf{u}, \end{aligned}$$

since the second term is zero by the envelope theorem.

Given any $\pi > \mu$, $\hat{V}(\pi, \mu)$ is the supremum of linear functions, i.e.

$$\hat{V}(\pi, \mu) = \sup_b \{(\pi - \mu)\mathbf{u} \cdot (\mathbf{p}_{bG} - \mathbf{p}_{bB})\},$$

and is thus convex in π. Since V equals the maximum of $\hat{V}$ and 0, it is also convex in π.

[24]If $\hat{b}(\mu) = 1$, then equation (9) applies to the left hand derivative of $c_2(b)$ at 1.

Proof of Lemma 2

Write the difference in beliefs as

$$\begin{aligned}\pi_b^k - \mu_{b^*}^k &= \frac{\lambda p_{bG}^k}{p_{b\lambda}^k} - \frac{\lambda p_{b^*G}^k}{p_{b^*\lambda}^k} \\ &= \frac{\lambda}{p_{b\lambda}^k p_{b^*\lambda}^k}\left(p_{bG}^k p_{b^*\lambda}^k - p_{b^*G}^k p_{b\lambda}^k\right).\end{aligned}$$

Using the fact that p_{bG}^k is a convex combination of p_{1G}^k and p_{0G}^k (and similarly $p_{b^*G}^k, p_{b^*\lambda}^k$ and $p_{b\lambda}^k$), this can be re-written as

$$\pi_b^k - \mu_{b^*}^k = \frac{\lambda(1-\lambda)}{p_{b\lambda}^k p_{b^*\lambda}^k}(b^* - b)\left[p_{0G}^k p_{1B}^k - p_{0B}^k p_{1G}^k\right]. \tag{10}$$

For any $b < b^*$,the sign of $\pi_b^k - \mu_{b^*}^k$ depends only on the sign of $p_{0G}^k p_{1B}^k - p_{0B}^k p_{1G}^k$, i.e., on the sign of $\ell_0^k - \ell_1^k$, thereby proving the lemma.

Appendix B: Proofs related to Theorem 2

Proof of Lemma 4

Let $\tilde{V}(\pi,\mu)$ denote the agent's payoff when he accepts the job and chooses b_μ:

$$\tilde{V}(\pi,\mu) = (\pi - \mu)(\mathbf{p}_{b_\mu G} - \mathbf{p}_{b_\mu B}) \cdot \mathbf{u}_\mu. \tag{11}$$

To prove the lemma, it clearly suffices to show that $\tilde{V}(\pi,\mu)$ is non zero when $\pi \neq \mu$. That is, we need to show that generically,

$$(\mathbf{p}_{bG} - \mathbf{p}_{bB}) \cdot \mathbf{u}_\mu = (1-b)(\mathbf{p}_{0G} - \mathbf{p}_{0B}) \cdot \mathbf{u}_\mu + b(\mathbf{p}_{1G} - \mathbf{p}_{1B}) \cdot \mathbf{u}_\mu \tag{12}$$

is not equal to zero at $b = b_\mu$. Since (12) is an affine function of b, it equals zero at most one value of b or at every value of b.

If (12) is zero at every value of b, this implies

$$(\mathbf{p}_{0G} - \mathbf{p}_{0B}) \cdot \mathbf{u}_\mu = 0$$

and

$$(\mathbf{p}_{1G} - \mathbf{p}_{1B}) \cdot \mathbf{u}_\mu = 0.$$

We now show that this violates Assumption 1* when output signals are binary, and will not be satisfied generically when $K > 2$. Observe that since $b_\mu > 0$, $\mathbf{u}_\mu$ is not a constant vector.

Let p and q be arbitrary probability distributions on Y, so that $p, q \in \Delta^{K-1}$, the $K-1$ dimensional simplex. Given a vector $\mathbf{u} \in \mathbb{R}^K$, let $\tilde{\mathbf{u}}$ denote the $K-1$

dimensional vector $(u^k - u^1)_{k=2}^K$. Similarly, let $\tilde{\mathbf{p}} = (p^k)_{k=2}^K$ and $\tilde{\mathbf{q}} = (q^k)_{k=2}^K$. It is easy to verify that

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{u} = (\tilde{\mathbf{p}} - \tilde{\mathbf{q}}) \cdot \tilde{\mathbf{u}},$$

which allows us to write

$$(\mathbf{p}_{0G} - \mathbf{p}_{0B}) \cdot \mathbf{u}_\mu = (\tilde{\mathbf{p}}_{0G} - \tilde{\mathbf{p}}_{0B}) \cdot \tilde{\mathbf{u}}_\mu. \tag{13}$$

and

$$(\mathbf{p}_{1G} - \mathbf{p}_{1B}) \cdot \mathbf{u}_\mu = (\tilde{\mathbf{p}}_{1G} - \tilde{\mathbf{p}}_{1B}) \cdot \tilde{\mathbf{u}}_\mu. \tag{14}$$

Consider first the case of binary signals, i.e., $K = 2$. In this case, each of the above probability vectors (e.g. $\tilde{\mathbf{p}}_{0G}$) are one-dimensional scalars, and $\tilde{\mathbf{u}}_\mu$ is a non-zero real number, since it must provide incentives for positive effort. Assumption 1* then ensures that both (13) and (14) cannot be zero.

Now consider $K > 2$. Since $\tilde{\mathbf{u}}_\mu$ is not equal to the null vector, the set of $(\tilde{\mathbf{p}}_{0G} - \tilde{\mathbf{p}}_{0B})$ values such that $(\tilde{\mathbf{p}}_{0G} - \tilde{\mathbf{p}}_{0B}) \cdot \tilde{\mathbf{u}}_\mu = 0$ defines a hyperplane in $\mathbb{R}^{K-1}$, that is of Lebesgue measure zero in $\mathbb{R}^{K-1}$. Similarly, the set of values of $(\tilde{\mathbf{p}}_{1G} - \tilde{\mathbf{p}}_{1B})$ such that $(\tilde{\mathbf{p}}_{1G} - \tilde{\mathbf{p}}_{1B}) \cdot \tilde{\mathbf{p}}_\mu = 0$ also lies in the same hyperplane. Since the set of distributions $(\tilde{\mathbf{p}}_{1G}, \tilde{\mathbf{p}}_{1B}, \tilde{\mathbf{p}}_{0G}, \tilde{\mathbf{p}}_{0B})$ such that $(\tilde{\mathbf{p}}_{1G} - \tilde{\mathbf{p}}_{1B})$ and $(\tilde{\mathbf{p}}_{0G} - \tilde{\mathbf{p}}_{0B})$ lie in the same hyperplane is of Lebesgue measure zero in $\mathbb{R}^{K-1}$, we conclude that the both (13) and (14) cannot be zero generically. This establishes that for generic information structures, (12) cannot be zero at every value of b.

If there is no value of b such that $(\mathbf{p}_{bG} - \mathbf{p}_{bB}) \cdot \mathbf{u}_\mu = 0$, the lemma is proved. So let $\breve{b}$ denote the single value of b such that $(\mathbf{p}_{eG} - \mathbf{p}_{e_B}) \cdot \mathbf{u}_\mu = 0$. Let $\varphi(b, \mu)$ denote the expected wage cost to the principal of inducing effort b at belief μ. We now show that for generic values of the vector $\mathbf{y}$, $b_\mu \neq \breve{b}$. $\breve{b}$ is a zero of the function of b defined by the right-hand side of (12), and does not depend upon the values of $\mathbf{y}$, while b_μ is defined by the condition

$$\sum_{k=1}^{K} (p_{1\mu}^k - p_{0\mu}^k) y^k = \varphi_1(b_\mu, \mu).$$

Fix an information structure $\mathbf{p}$: this determines $\breve{b}$. This determines also determines the minimum cost of inducing any effort level, $\varphi(b, \mu)$. Since the left-hand side of the equation defining b_μ is linear in $\mathbf{y}$, $b_\mu \neq \breve{b}$ for almost all values of $\mathbf{y}$. This completes the proof.

Proof of Lemma 5

Suppose that $\frac{p_{1G}^k}{p_{0G}^k} = \frac{p_{1B}^k}{p_{1B}^k}$ for all $k \in \{1, 2, ..K\}$. Let θ denote this common ratio. Fix an arbitrary signal, say K. Since $p_{\omega e}^K = 1 - \sum_{k=1}^{K-1} p_{e\omega}^k$,

$$\frac{1-\sum_{k=1}^{K-1} p_{1G}^k}{1-\sum_{k=1}^{K-1} p_{0G}^k} = \frac{1-\sum_{k=1}^{K-1} p_{1B}^k}{1-\sum_{k=1}^{K-1} p_{0B}^k}.$$

Multiplying by the two denominators and simplifying, we get

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k - \left(\sum_{k=1}^{K-1} p_{1G}^k\right)\left(\sum_{k=1}^{K-1} p_{0B}^k\right) = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k - \left(\sum_{k=1}^{K-1} p_{1B}^k\right)\left(\sum_{k=1}^{K-1} p_{0G}^k\right).$$

Using the fact that $p_{1G}^k p_{0B}^k = p_{1B}^k p_{0G}^k$ for $k \in \{1, 2.., K-1\}$, we get

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k - \left(\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1G}^k p_{0B}^j\right) = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k - \left(\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1B}^k p_{0G}^j\right).$$

By rewriting p_{1G}^k as θp_{1B}^k and p_{0G}^j as θp_{0B}^j, we get

$$\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1G}^k p_{0B}^j = \sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} \theta p_{1B}^k p_{0B}^j,$$

$$\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1B}^k p_{0G}^j = \sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} \theta p_{1B}^k p_{0B}^j,$$

so that the above two expressions are equal. Hence

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k,$$

which implies

$$p_{1G}^K + p_{0B}^K = p_{1B}^K + p_{0G}^K.$$

But since $p_{1G}^K p_{0B}^K = p_{1B}^K p_{0G}^K$, the two together imply either $p_{1G}^K = p_{1B}^K$ and $p_{0B}^K = p_{0G}^K$ or $p_{1G}^K = p_{0G}^K$ and $p_{0B}^K = p_{1B}^K$. Since the choice of signal K was arbitrary, this is true for every signal, contradicting Assumption 1*.

Proof of Theorem 2

The expected second period continuation value, $W(a, a^*)$, can be written as:

$$W(a, a^*) = \begin{cases} \sum_{y^k \in Y^D} p_{a\lambda}^k V(\pi_a^k, \mu_{a^*}^k) \text{ if } a < a^* \\ \sum_{y^k \in Y^U} p_{a\lambda}^k V(\pi_a^k, \mu_{a^*}^k) \text{ if } a > a^*. \end{cases}$$

The left-hand derivative of $W(a, a^*)$ at $a = a^* \in (0, 1)$, is given, after some simplification, by :

$$\left.\frac{\partial W^-(a,a^*)}{\partial a}\right|_{a=a^*} = \sum_{y^k\in Y^D} p^k_{a^*\lambda} V^+_\pi(\pi^k_{a^*},\mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*}$$

which is strictly negative if Y^D is non-empty.

Similarly, the right-hand derivative is given by

$$\left.\frac{\partial W^+(a,a^*)}{\partial a}\right|_{a=a^*} = \sum_{y^k\in Y^U} p^k_{a^*\lambda} V^+_\pi(\pi^k_{a^*},\mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*},$$

and this is strictly positive if Y^U is non-empty. Since either Y^U or Y^D is non-empty, we conclude that $\left.\frac{\partial W^+(a,a^*)}{\partial a}\right|_{a=a^*} > \left.\frac{\partial W^-(a,a^*)}{\partial a}\right|_{a=a^*}$.

The rest of the argument is identical to that in the proof of Theorem 1.

Appendix C: Proofs related to Theorem 3

Proof of Lemma 6

A menu is globally incentive compatible if and only if for any μ and $\hat{\mu}$,

$$V(\mu,\mu) - V(\mu,\hat{\mu}) \geq 0,$$

where

$$V(\mu,\hat{\mu}) = \Delta(\hat{\mu})[b(\mu,\Delta(\hat{\mu}))M(\mu) + \mu p_{0G} + (1-\mu)p_{0B}] + u_L(\hat{\mu}) - c_2(b(\mu,\Delta(\hat{\mu})))$$

is the payoff of belief-type μ choosing contract $\Delta(\hat{\mu})$ and choosing effort **optimally**.

Rewrite the incentive compatibility condition as

$$V(\mu,\mu) - V(\hat{\mu},\hat{\mu}) \geq V(\mu,\hat{\mu}) - V(\hat{\mu},\hat{\mu}).$$

Using condition (4) we can rewrite the left hand side of the inequality as

$$V(\mu,\mu) - V(\hat{\mu},\hat{\mu}) = \int_{\hat{\mu}}^{\mu} \Delta(z)(\rho b(z,\Delta(z)) + p_{0G} - p_{0B})dz.$$

The right hand side of the inequality is

$$\begin{aligned} V(\mu,\hat{\mu}) - V(\hat{\mu},\hat{\mu}) = \\ &\Delta(\hat{\mu})\left[(\mu-\hat{\mu})(p_{0G}-p_{0B}) + b(\mu,\Delta(\hat{\mu}))M(\mu) - b(\hat{\mu},\Delta(\hat{\mu}))M(\hat{\mu})\right] \\ &+ c_2(b(\hat{\mu},\Delta(\hat{\mu}))) - c_2(b(\mu,\Delta(\hat{\mu}))). \end{aligned}$$

Using (3) and integrating by parts, we can replace the difference in costs:

$$c_2(b(\mu,\Delta(\hat{\mu}))) - c_2(b(\hat{\mu},\Delta(\hat{\mu}))) =$$
$$\int_{\hat{\mu}}^{\mu} c_2'(b(z,\Delta(\hat{\mu})))b_1(z,\Delta(\hat{\mu}))dz =$$
$$\int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})M(z)b_1(z,\Delta(\hat{\mu}))dz =$$
$$\Delta(\hat{\mu})[b(\mu,\Delta(\hat{\mu}))M(\mu) - b(\hat{\mu},\Delta(\hat{\mu}))M(\hat{\mu})] - \int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})M(z)b(z,\Delta(\hat{\mu}))\rho dz.$$

Thus, incentive compatibility can be rewritten as the following set of inequalities:

$$\forall \mu,\hat{\mu} : \int_{\hat{\mu}}^{\mu} \Delta(z)[\rho b(z,\Delta(z)) + p_{0G} - p_{0B}]dz \geq \int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})[\rho b(z,\Delta(\hat{\mu})) + p_{0G} - p_{0B}]dz. \quad (15)$$

$b(\mu,\Delta)$ is increasing in Δ. Hence if $\rho \geq 0$, the collection of inequalities (15) holds if and only if $\Delta(\mu)$ is increasing in μ.

Proof of Lemma 7

The partial derivatives of $\tilde{b}_y$ are both negative:

$$\frac{\partial \tilde{b}_y(a,h)}{\partial a} = \frac{[2y_h M_y - c_2'(\tilde{b}_y)]\rho\mu_y'}{M_y c_2''(\tilde{b}_y) - [c_2''(\tilde{b}_y)(1+\rho) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c_2'''(\tilde{b}_y)]\mu_y' h} \leq 0$$
$$\frac{\partial \tilde{b}_y(a,h)}{\partial h} = \frac{[c_2'(\tilde{b}_y) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c_2''(\tilde{b}_y)]\mu_y'}{M_y c_2''(\tilde{b}_y) - [c_2''(\tilde{b}_y)(1+\rho) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c_2'''(\tilde{b}_y)]\mu_y' h} \leq 0.$$

Proof of Lemma 8

We begin with the observation that if $h(a)$ is increasing, then, by Lemma 7, $\tilde{b}_y(a,h(a))$ is decreasing in a (or, increasing in $\mu_y(a)$).

Note that

$$\Delta(\mu_y(a)) = \frac{y_h[M_y(a)]^2 + \mu_y'(a)h(a)c_2''(\tilde{b}_y(a,h(a)))(\rho\tilde{b}_y(a,h(a)) + p_{0G} - p_{0B})}{1 + \rho\frac{-\mu_y'(a)h(a)}{M_y(a)}}.$$

The denominator of this expression is decreasing and the numerator is increasing in $\mu_y(a)$.

Proof of Lemma 9

Equation (8) is a Volterra equation of the first kind in the Urysohn form with a degenerate linear kernel (Polyanin and Manzhirov, 2008, p. 676). Let

$$v_y(a,b) = \frac{\mu_y'(a)}{M_y(a)} c_2'(b)(p_{0G} - p_{0B} + b\rho),\ y \in \{L, H\}.$$

The partial derivatives of this function are

$$\begin{aligned} v_{y1}(a,b) &= \frac{\partial}{\partial a} v_y(a,b) = \frac{\mu_y''(a) M_y(a) - \rho[\mu'(a)]^2}{[M_y(a)]^2} c_2'(b)(p_{0G} - p_{0B} + b\rho), \\ v_{y2}(a,b) &= \frac{\partial}{\partial b} v_y(a,b) = \frac{\mu_y'(a)}{M_y(a)} (p_{0G} - p_{0B} + b\rho) c_2''(b) + \rho c_2'(b) \le 0. \end{aligned}$$

Also let $\mathcal{P}(a) = \Pr\{H \mid a\}$, so that $\mathcal{P}'(a) = \lambda\rho + p_{1B} - p_{0B}$.

By differentiating equation (8) twice we rewrite it as a first order ODE:

$$\begin{aligned} c_1'' = v_{L1} &+ v_{L2}(\tilde{b}_{L1} + \tilde{b}_{L2} h') + 2\mathcal{P}'(v_H - v_L) \\ &+ \mathcal{P}(v_{H1} - v_{L1} + v_{H2}(\tilde{b}_{H1} + \tilde{b}_{H2} h') \\ &- v_{L2}(\tilde{b}_{L1} + \tilde{b}_{L2} h')). \end{aligned}$$

Equivalently,

$$h' = \frac{c_1'' - \Big[2\mathcal{P}'(v_H - v_L) + \mathcal{P}(v_{H1} + v_{H2}\tilde{b}_{H1}) + (1-\mathcal{P})(v_{L1} + v_{L2}\tilde{b}_{L1})\Big]}{\mathcal{P} v_{H2}\tilde{b}_{H2} + (1-\mathcal{P}) v_{L2}\tilde{b}_{L2}}. \tag{16}$$

Recall that $\tilde{b}_y$ is not differentiable when $h = \tilde{h}$ (consequently, h is not differentiable at that point too). To work around this issue, we rewrite the equation (16) as two equations, and use the left-hand derivative of $\tilde{b}$ which always exists and is equal to the derivative of $\beta(a,h)$:

$$h_1' = \frac{c_1'' - [2\mathcal{P}' v_H + \mathcal{P}(v_{H1} + v_{H2}\beta_{H1})]}{\mathcal{P} v_{H2}\beta_{H2}},\ \tilde{h}_L \le h_1 \le \tilde{h}_H, \tag{17}$$

$$h_2' = \frac{c_1'' - [2\mathcal{P}'(v_H - v_L) + \mathcal{P}(v_{H1} + v_{H2}\beta_{H1}) + (1-\mathcal{P})(v_{L1} + v_{L2}\beta_{L1})]}{\mathcal{P} v_{H2}\beta_{H2} + (1-\mathcal{P}) v_{L2}\beta_{L2}},\ 0 \le h_2 \le \tilde{h}_L.$$

The right-hand sides for both equations are continuous in a and Lipschitz-continuous in h, therefore both equation have unique sollutions given their respective boundary conditions. These boundary conditions are

$$\begin{aligned} h_1(\bar{a}_\Omega) &= \tilde{h}_H(\bar{a}_\Omega), \\ h_2(\tilde{a}_\Omega) &= h_1(\tilde{a}_\Omega) = \tilde{h}_L(\tilde{a}_\Omega). \end{aligned}$$

To find the lowest level of effort possible in equilibrium we examine $h_2(a)$: if $h_2(a) > 0$ for any $a \geq 0$, then $\underline{a}_\Omega = 0$. Otherwise, $\underline{a}_\Omega$ solves $h_2(\underline{a}_\Omega) = 0$. In the former case, the distribution of efforts has a mass point at the lower bound of the support; otherwise, the distribution is continuous everywhere.

We obtain the solution for equation (8) by gluing the functions h_1 and h_2 together:

$$h_\Omega(a) = \begin{cases} h_1(a), & \text{if } a \in [\tilde{a}_\Omega, \overline{a}_\Omega], \\ h_2(a), & \text{if } a \in (\underline{a}_\Omega, \tilde{a}_\Omega), \\ 0, & \text{if } a = \underline{a}_\Omega. \end{cases}$$

Note that h is continuous everywhere except, perhaps, at $\underline{a}_\Omega$. It is also differentiable everywhere except at $\tilde{a}_\Omega$ and $\underline{a}_\Omega$.

Interestingly, Ω does not enter the differential equation, which means that the solution depends on Ω only through the boundary condition. This implies that the functions h_1 and h_2 are continuous in (a, Ω) jointly and so is $h(a)$ except perhaps at $a = \underline{a}_\Omega$.

Finally, we show that solution $h_\Omega(a)$ is increasing in a. First, observe that for any $a \in [0,1] : h_\Omega(a) \leq \tilde{h}_H(a)$ and $\overline{h} := \sup_{a\in[0,1]} \tilde{h}_H(a) < \infty$. Second, note that for any $a \in [0,1]$ and $h(a) \leq \tilde{h}_H(a)$

$$\mathcal{P} v_{H2} \tilde{b}_{H2} + (1 - \mathcal{P}) v_{L2} \tilde{b}_{L2} > 0.$$

Third and final, there exists κ such that for all $a \in [0,1]$ and $h \leq \overline{h}$:

$$\kappa \geq \left[2\mathcal{P}'(v_H - v_L) + \mathcal{P}(v_{H1} + v_{H2}\tilde{b}_{H1}) + (1 - \mathcal{P})(v_{L1} + v_{L2}\tilde{b}_{L1}) \right].$$

Therefore, if $c_1''(a) \geq \kappa$ for all $a \in [0,1]$, equation (16) implies that $h_\Omega'(a) \geq 0$. Importantly, the bound κ does not depend on the first period contract Ω.

Proof of Lemma 10

First, note that $\underline{a}_\Omega$ and $\overline{a}_\Omega$ are continuous in Ω and $\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega)$ depends on Ω only through $\underline{a}_\Omega$ (since $h_\Omega(\underline{a}_\Omega) = 0$), therefore is continuous in Ω as well.

Second, because $1/h_\Omega(a)$ is continuous in Ω for any a except, perhaps, $a = \underline{a}_\Omega$, $G_\Omega(a)$ is continuous in Ω for any a and $g_\Omega(a)$ is continuous in Ω everywhere except perhaps at $a = \underline{a}_\Omega$ (see Jeffery, 1925). Therefore, $G_\Omega(\underline{a}_\Omega)[\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega)]$ is continuous in Ω.

Third, note that $\tilde{b}_y(a, h_\Omega(a))$ is continuous in Ω for any a except, perhaps, $a = \underline{a}_\Omega$, and therefore $[\pi_1(a) + \pi_2(a, \Omega)]g_\Omega(a)$ is continuous in Ω for every a except, perhaps $a = \underline{a}_\Omega$. Using Jeffery (1925), we can conclude that $\int_0^1 [\pi_1(a) + \pi_2(a, \Omega)]g_\Omega(a)da$ is continuous in Ω.