The Ratchet Effect: A Learning Perspective*

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Abstract

We examine the ratchet effect under moral hazard and symmetric learning by worker and firm about new technology. Shirking increases the worker's future payoffs, since the firm overestimates job difficulty. High-powered incentives to deter shirking induce the agent to over-work, since he can quit if the firm thinks the job is too easy. With continuous effort choices, no deterministic interior effort is implementable. We provide conditions under which randomized effort is implementable, so that a profit-maximizing distribution over efforts exists.

Keywords: ratchet effect, moral hazard, learning, randomized effort.

JEL codes: D83, D86.

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1 Introduction

The ratchet effect is one of the earliest problems noted by modern incentive theory, and arose in Soviet planning (Berliner, 1957). If the factory met or exceeded its plan target, the target for subsequent years was increased, reducing current effort incentives for the manager (Weitzman, 1980). The problem also arises in capitalist firms, as Milgrom and Roberts (1992) note. When a firm installs new equipment, firms and workers have to learn what is the appropriate work standard. It is efficient to use future information to adjust the standard. But this reduces work incentives today.[1] The ratchet effect arises in marketing, when salesmen are paid bonuses that depend exceeding a quota, and the quota is adjusted based on past performance. It also arises in a regulatory context, where both the regulator and the firms are uncertain about the effects of new technology (see Meyer and Vickers, 1997).

The ratchet effect is a plausible explanation for the slow adoption and diffusion of new technology (see Geroski, 2000) and new management practices (see Bloom and Van Reenen, 2007, 2010). While Bloom and van Reenen argue that product market competition facilitates innovation and diffusion, agency problems and conflicts of interest within the firm are also likely to be an important factor, if harder to document. Atkin et al. (2017) conduct a field experiment that examines the adoption of a new innovation among soccer ball producers in Pakistan. Firms in the treatment group were provided with an innovation that was demonstrably superior in reducing non-labor inputs. Although the sector caters to the export market and is subject to serious competitive pressure, very few firms adopted the new technology. Atkin et al. (2017) argue that agency problems within the firm – specifically, a form of ratchet effect – are the most plausible explanation for this failure.

The goal of this paper is to study the ratchet effect in the context of new technology, where both workers and firms are uncertain about its efficacy. Our model differs from existing theoretical work on the ratchet effect, which mainly assumes that the agent already has private information. This literature studies dynamic mechanism design without commitment, and examines how the principal induces the agent to reveal her private information.

We study a model where both employer and worker are learning about the technology, with ex ante symmetric information. Since the technology is new, it is plausible that the worker does not already have private information, and if he does acquire private information, this takes time. One must allow for contractual remedies that could address this at the outset. In hidden information environments, it is well known that contracting at the ex ante stage, before the agent has private information, is more efficient than ex post contracting. If the agent is risk neutral, ex ante contracting

[1] Mathewson (1931),Roy (1952) and Edwards (1979) are workplace studies that document the importance of "output restriction" Interestingly, they find that workers collectively enforce norms of lower output, and sanction individuals who break the norm, highlighting the limitations of yardstick competition in overcoming the ratchet effect.

enables full efficiency (see e.g., Laffont and Martimort, 2009). One might expect similar results in our setting. Milgrom and Roberts (1992) present an illuminating (albeit somewhat informal) discussion of the ratchet effect in a learning context. They assume that output is the sum of project quality and effort, plus a normally distributed shock. They argue that the ratchet effect implies that incentives need to be more high powered at the beginning of the relationship, thereby subjecting the worker to additional risk. Long term commitments alleviate these inefficiencies, but it may be hard to stick to these commitments since they are likely to be inefficient ex post. While their discussion is extremely insightful, they do not make explicit their assumptions. Our formal analysis shows that the problem may be substantially more severe than that envisaged by Milgrom and Roberts (1992), since it is impossible to incentivize *any* interior effort level with certainty. When targeting an effort level, the best the principal can do is incentivize the agent to choose an effort randomly within certain range. Such a contract necessarily induces inefficiency, in both periods of the interaction.

More specifically, we study optimal contracting in a two-period model where firm and worker are ex ante symmetrically uncertain about the difficulty of the job. To focus on the ratchet effect, we assume that the firm (or principal) cannot commit to long term contracts, but chooses short-term contracts optimally. Since there is no limited liability and the principal has all the bargaining power, the agent will not be paid any more than his outside option. Furthermore, since uncertainty pertains to the nature of the job, the outside option does not depend upon what is learnt regarding the job. The ratchet effect arises since the agent can manipulate the beliefs of the principal by shirking. Assume that the principal seeks to induce an interior effort level a^*. Suppose that the agent shirks and chooses $a < a^*$, and some output level y is realized, that is publicly observed by both parties. The agent's beliefs regarding job difficulty will differ from that of the principal, since the principal updates her beliefs on observing y assuming that a^* has been chosen while the agent knows that a has chosen. Since we allow a fairly general information structure, the agent may be more optimistic or more pessimistic than the principal, for a given output realization y. If the agent becomes more pessimistic than the principal, he incurs no loss – the job only pays him his reservation utility in equilibrium, and he can quit if he earns less. If he becomes more optimistic than the principal, he earns a rent. Under fairly general assumptions, we show that there must always be some output level such that the agent is more optimistic than the principal when he shirks. Thus the agent raises his continuation value by shirking a little, and incentives have to be high powered in order to deter shirking. However, if the principal provides high powered incentives, this makes it profitable for the agent to over-work, i.e. to choose $a > a^*$, since by doing so, he increases his current payoff. This may cause the job to be more unattractive tomorrow, but since the agent is always free to quit, he incurs no loss in consequence. Thus, at any interior effort level a^*, the left hand derivative of the agent's continuation value function is negative, while the right hand derivative is non-negative. This *convex kink* implies that the first order conditions for implementing a^* can never be satisfied, no matter how convex the cost of effort. The underlying conceptual reason reason for the negative result is that we have an agency model where the agent makes choices from a continuum set (effort) as well as a discrete choice set (participation), so that the standard envelope theorem does not hold.

Our impossibility result raises a natural question: what can principal do apart from inducing extremal efforts? To study this question, which is complex, we invoke several simplifying assumptions.[2] Suppose that the principal chooses a level of first period incentive pay Ω that is intermediate, so that extremal efforts are not optimal. We show that the continuation game following Ω has an equilibrium where the agent randomizes over efforts in an interval, $[\underline{a}_\Omega, \overline{a}_\Omega]$, according to the distribution G. Following an effort a and output realization y, the agent updates his private beliefs about the difficulty of the job to $\mu_y(a)$. Therefore, from the point of view of the principal, the beliefs of the agent are distributed on an interval $[\mu_y(\overline{a}_\Omega), \mu_y(\underline{a}_\Omega)]$, with distribution $F_{\Omega,y}$. In the second period, the principal faces a screening problem, where the agent has private information and is also subject to moral hazard.[3] This allows the agent to earn informational rents which are monotonically decreasing in first period effort; the highest rents are earned by the lowest effort level $\underline{a}_\Omega$, while the highest first period effort $\overline{a}_\Omega$ earns zero rents. Thus, the highest effort level $\overline{a}_\Omega$ maximizes the first period payoff of the agent given Ω, while "shirking" is costly, given the level of incentive pay, and compensated via an expected informational rent $W(a)$ that ensures the agent's indifference between all efforts a in the interval $[\underline{a}_\Omega, \overline{a}_\Omega]$. Randomized effort results in two distortions, since first period and second period efforts are not chosen optimally, with the latter being distorted downwards due to private information. Finally, we show that the principal's overall payoff is a continuous function of Ω, so that a profit-maximizing first period contract exists. We also provide an illustrative numerical example. Since both extremal efforts and random effort have negative efficiency consequences, our results imply that the ratchet effect can reduce benefits from introducing a new technology.

1.1 Related literature

Previous work on the ratchet effect differs from our analysis, since it usually assumes that the worker has private information at the outset. Lazear (1986) argues that high powered incentives are able to overcome the ratchet effect, without any efficiency loss, assuming that the worker is risk neutral. Gibbons (1987) shows that Lazear's result depends upon an implicit assumption of long term commitment; in its absence, one cannot induce efficient effort provision by the more productive type. Laf-

[2] We assume: the agent is risk-neutral; output is binary; effort is weakly more productive in the good state; the information structure satisfies uniform optimism, so that the agent is more optimistic than the principal after both signal realizations.

[3] Note that the agent's private information is directly payoff-relevant for the principal.

font and Tirole (1988) prove a general result, that one cannot induce full separation given a continuum of types.[4] Laffont and Tirole (1993) have a comprehensive discussion, and consider both the case of binary types and of a continuum of types. We will relate our findings to their discussion after presenting our main result (Theorem 1), in section 2.3.[5]

Formally, this paper is more related to models of dynamic moral hazard with learning and ex ante symmetric uncertainty, as pioneered by Holmström (1999), in his career concerns model. Extensions of the career concerns model include Gibbons and Murphy (1992), Dewatripont et al. (1999) and Bonatti and Hörner (2017). Recently, there has been increased interest in agency models with learning, where the uncertainty pertains to the nature of the project rather than the talent of the agent. Bergemann and Hege (1998, 2005), Manso (2011), Hörner and Samuelson (2015) and Kwon (2011) analyze agency models with binary effort and binary signals. DeMarzo and Sannikov (2016) , Cisternas (2018) and Prat and Jovanovic (2014) study continuous time agency models with continuum action spaces.[6] From a methodological point of view, a crucial difference is that our paper allows both continuous choices (in the effort dimension) and discrete choices (regarding participation), whereas other papers allow either only discrete or only continuous choices – see Subsection 2.3 for a further discussion.

Our results are also reminiscent of papers that find that pure strategy equilibria do not exist in one-period models without commitment. Fudenberg and Tirole (1990) examine classical moral hazard with a risk averse agent, where the principal cannot commit not to renegotiate the contract at the interim stage – after effort is chosen, but before output is realized. They show that a pure strategy equilibrium does not exist, and characterize mixed strategy equlibria. Gul (2001) and González (2004) study the hold up problem when investment is unobserved, and find that effort must be random. Bhaskar and Roketskiy (2021) analyze dynamic non-linear pricing with non-separable preferences and show that similar issues arise. Solving for a mixed strategy equilibrium is more complex in our context, since the second period payoff of the agent after any first period effort is random, and depends upon the output realization. Furthermore, the second period problem combines adverse selection and moral hazard, whereas only adverse selection is present in the previous work, since effort is sunk.

The rest of this paper is as follows. Section 2 sets out our model of the ratchet

[4]Malcomson (2016) shows that the no full-separation result also obtains in a relational contracting setting, where the principal need not have all the bargaining power, as long as continuation play following full separation is efficient.

[5]Freixas et al. (1985) and Carmichael and MacLeod (2000) are other important papers on the ratchet effect that employ the private information paradigm.

[6]Bhaskar and Mailath (2019) analyze a model of the ratchet effect with binary effort. They examine how the costs of incentivizing high effort vary with the length of the interaction, and show that the difficulty of the incentive problem increases at least linearly with time horizon, so that inducing high effort consistently becomes unprofitable.

effect, and presents the impossibility result. Section 3 analyzes random effort, and provides conditions under which an equilibrium with random effort exists.

2 The Model

Our model combines moral hazard with uncertainty regarding job difficulty. There are two states of the world $\omega \in \{B, G\}$, with the job being good (easy) in G, and bad (hard) in B, with $\lambda \in (0,1)$ denoting the common prior that $\omega = G$. The uncertainty concerns how difficult it is to succeed on *this* job. Importantly, learning does *not* affect the outside option of the agent, which is fixed, and normalized to 0. The agent chooses effort $e \in [0,1]$, at cost $c(e)$, which is increasing, strictly convex and differentiable. The agent learns knowing his own effort choice and a realized public signal, $y \in Y$, where $Y := \{y^1, y^2, \ldots, y^K\}$ is a finite set of signals. The principal learns knowing only the signal, since the agent's effort is not public (i.e., it is not observed by the principal). The agent's flow utility from a wage payment $w \in \mathbb{R}$ and choosing $e \in [0,1]$ is

$$v(w) - c(e),$$

where v is strictly increasing and strictly concave (so the agent is risk averse) – we show in Subsection 2.5 that our analysis extends to the case where the agent is risk neutral. We assume that wage payments are unrestricted, i.e. there is no limited liability.

A spot contract specifies the wage payment as a function of the realized signal. It is more convenient to work with gross utility schedules, so we write a spot contract as $\boldsymbol{u} := (u^1, \ldots, u^K)$, where u^k is the gross utility the agent will receive after signal y^k. Let $w(.) = v^{-1}(.)$ denote the inverse function corresponding to $v(.)$.

The principal is risk neutral and her flow utility is $y - w(u)$. In each period, the principal makes a take-it-or-leave-it offer of a spot contract to the agent. If the agent refuses, the relationship is dissolved and the game ends.

The probability of signal y^k at action e and state $\omega \in \{B, G\}$ is $p^k_{e\omega}$.

Consider first the extremal efforts, $e \in \{0,1\}$. We will assume that a "high" signal is both a signal of the good state and of high effort. We capture this by the following assumption.

Assumption 1.

1. *There exists an* informative *signal, i.e.,* $\exists y^k \in Y$ *such that* $p^k_{0B} \neq p^k_{1G}$. *For any informative signal* $y^k \in Y$,

$$\min\left\{p^k_{0B}, p^k_{1G}\right\} < p^k_{0G} < \max\left\{p^k_{0B}, p^k_{1G}\right\},$$

and

$$\min\left\{p^k_{0B}, p^k_{1G}\right\} < p^k_{1B} < \max\left\{p^k_{0B}, p^k_{1G}\right\}.$$

2. *Signals have full support:* $p^k_{e\omega} > 0$ *for all* k, e, ω.

Partition the set of signals into a set of "high" signals Y^H, "low" signals Y^L, and neutral $Y \setminus (Y^H \cup Y^L)$ by defining

$$y^k \in Y^H \iff p^k_{1G} > p^k_{0B}$$

and

$$y^k \in Y^L \iff p^k_{1G} < p^k_{0B}.$$

Let μ denote the probability assigned to state G. The probability of signal y^k at effort level $e \in \{0,1\}$ and belief μ is $p^k_{e\mu} = \mu p^k_{eG} + (1-\mu)p^k_{eB}$. Assumption 1 implies that for any informative signal y^k

$$y^k \in Y^H \iff p^k_{1G} > p^k_{1B}, p^k_{0G} > p^k_{0B} \iff p^k_{1\mu} > p^k_{0\mu}$$

and

$$y^k \in Y^L \iff p^k_{1G} < p^k_{1B}, p^k_{0G} < p^k_{0B} \iff p^k_{1\mu} < p^k_{0\mu}.$$

In other words, we can partition the signal space into high, low and neutral signals. High (resp. low) signals arise with higher (resp. lower) probability when either the agent exerts effort *or* the state is good. With binary signals (taking values H or L), where $p^H_{1G} > p^H_{0B}$, this assumption requires that p^H_{0G} and p^H_{1B} both belong to the interval (p^H_{1G}, p^H_{0B}).

Our second assumption extends the information structure to all effort levels in $[0,1]$. With a continuum of effort levels, we need to employ the first-order approach to solve for the optimal contract, even in the static case. We therefore assume the Hart and Holmström (1987) sufficient conditions for the validity of this approach, and our assumption is an adaptation of their conditions.

Assumption 2. *For any* $y^k \in Y$, *and any* $\omega \in \{G, B\}$, $p^k_{e\omega} = ep^k_{1\omega} + (1-e)p^k_{0\omega}$.[7]

We assume that the principal and agent interact for two periods – two periods suffice to make the main points of our paper. The agent discounts future payoffs at rate $\delta \in (0,1]$. The principal's discount factor is possibly different, but will play little role in the analysis. A key assumption is the absence of inter-temporal commitments – neither the principal nor the agent can commit in period one regarding the contract in period two. This implies that payments must satisfy incentive compatibility and individual rationality period by period. We denote the first period effort by a and the second period effort by b.

[7]Hart and Holmström (1987) assume a linear cost of effort and that the probability of y^k is a convex combination of two distributions, a "good" one and a "bad". They assume that the weight on the good distribution is an increasing and concave function of effort. To see that our parameterization is equivalent to theirs, define a new effort variable, $c(e)$. This gives linear costs and a concave weighting function.

We study the dynamic game induced by this contracting problem, and solve for perfect Bayesian equilibria that satisfy sequential rationality, with beliefs given by Bayes rule. Sequential rationality is required to ensure that the contract offered by the principal at $t = 2$ is optimal, and so the agent's participation constraint binds. Although differences in beliefs between the principal and the agent will play an important part in the analysis, we do not have to specify beliefs off the equilibrium path for the uninformed party (the principal). Since effort choice by the agent is private and public signals have full support, the principal does not see an out of equilibrium action, except when the game ends by the agent refusing the contract (at which point, beliefs are moot). Deviations by the uninformed party (the principal) have no implications for beliefs.

We focus on pure strategy equilibria, where the effort choice by the agent in period one is deterministic. If the agent chooses effort a^* at $t = 1$, and output y^k is realized, then the common belief of the principal and agent at $t = 2$ is denoted by $\mu^k_{a^*}$. Because the agent plays a pure strategy, the principal's second order beliefs are degenerate. Sequential rationality implies that the principal offers a profit maximizing contract at $t = 2$. We assume that the project is profitable at all beliefs at $t = 2$, so that the principal always induces the agent to participate.

2.1 The Final Period

Suppose that after observing the results of the first period, the principal and the agent agree on how difficult the task is. Let μ denote a common belief of the principal and the agent at the beginning of the second period. Let $\boldsymbol{u} = (u^k)_{k=1}^K$ denote a spot contract. Let $\boldsymbol{w(u)} := (w(u^k))_{k=1}^K$ denote the vector of wages associated with $\boldsymbol{u}$. Let $\boldsymbol{y} = (y_k)_{k=1}^K$, and let the vector $\boldsymbol{p}_{b\mu} := (p^k_{b\mu})_{k=1}^K$ denote the probability distribution over Y given the second period effort b and belief μ.

Definition 1. *Effort $\hat{b}$ is* implementable *at $t = 2$ if there exists a spot contract $\boldsymbol{u}$ such that $\hat{b}$ is optimal for the agent under belief μ.*

Definition 2. *A period 2 contract $(\hat{b}(\mu), \boldsymbol{u}_\mu)$ is optimal given belief μ if it maximizes the principal's profits $\boldsymbol{p}_{b\mu}.(\boldsymbol{y} - \boldsymbol{w(u)})$ over all $(b, \boldsymbol{u})$ such that b is implementable.*

Claim 1. *In the final period, for any public belief μ, every effort $b \in [0, 1]$ is implementable. The profit maximizing contract , $(\hat{b}(\mu), \boldsymbol{u}_\mu)$, satisfies the first order conditions for the principal's maximization problem, and the agent's individual rationality constraint binds. The agent's incentive constraint binds if $b > 0$. For any contract $\boldsymbol{u}$, there is a unique effort level that maximizes the agent's utility.*

We omit the proof the first two parts of this claim, since it is almost identical to the argument in Hart and Holmström (1987). To prove the final part, note that the agent's payoff from choosing b equals $\boldsymbol{p}_{b\mu}.\boldsymbol{u} - c(b)$. Since the $\boldsymbol{p}_{b\mu}$ is linear in b, and $c(b)$ is strictly convex, there is a unique solution to the agent's maximization problem.

Sequential rationality implies that the principal always chooses the optimal contract at $t = 2$, given any belief μ. Assume that for any μ, the effort induced by the principal, $\hat{b}(\mu)$, is non-zero – this assumption is a mild one if $c_b(0) = 0$.

We now analyze the agent's payoff in the final period when his belief is π and differs from the principal's belief μ. Recall that $\boldsymbol{u}_\mu$ denotes the optimal contract offered by the principal when her belief is μ. Let $\hat{V}(\pi, \mu) = \max_b (\boldsymbol{p}_{b\pi}.\boldsymbol{u}_\mu - c(b))$ denote the payoff to the agent, conditional on accepting the job and choosing effort optimally.

$\hat{V}$ is computed under the distribution $\boldsymbol{p}_{b\pi}$, reflecting the fact that the agent has the correct beliefs, since he knows his actual effort choice at $t = 1$. Since the agent will quit when he gets less than his outside option, let $V(\pi, \mu) = \max\{\hat{V}(\pi, \mu), 0\}$ denote his payoff given optimal participation. $V(\mu, \mu) = 0$ when $\pi = \mu$ since the agent's participation constraint binds under the optimal contract if he has the same beliefs. The following lemma summarizes the relevant properties of V, the most important property being that the agent gets rents if $\pi > \mu$. Intuitively, since the second period contract must provide incentives for positive effort, it must reward high signals. But under Assumption 1, this also rewards more optimistic beliefs.

Lemma 1. *$V(\pi, \mu) > 0$ if and only if $\pi > \mu$. $\hat{V}(\pi, \mu)$ is differentiable and $V(\pi, \mu)$ is convex in π.*

Proof. See Appendix A. □

2.2 The First Period

Suppose that the principal seeks to induce effort level a^* at $t = 1$, when both parties have common prior beliefs λ. If the agent deviates and chooses a different from a^*, then the principal and agent will have different second period beliefs after output y^k. The principal will have belief $\mu^k_{a^*}$, while the agent will have belief π^k_a. The expected second period continuation value of the agent from choosing a when the principal induces a^* equals

$$W(a, a^*) = \sum_{k=1}^{K} p^k_{a\lambda} V(\pi^k_a, \mu^k_{a^*}).$$

Each term under the summation sign is non-negative, since $V(\pi^k_a, \mu^k_{a^*}) \geq 0$, given that the agent can always quit when $\pi^k_a < \mu^k_{a^*}$. Thus $W(a, a^*)$ is strictly positive as long as there is some y^k such that $\pi^k_a > \mu^k_{a^*}$.

Consider downward deviations by the agent to $a < a^*$. There will be some set of signals where the agent is more optimistic than the principal, and another set where where he is more pessimistic. The following lemma shows that signals can be partitioned into these sets independently of the precise magnitudes of a and a^*. For

$a \in \{0,1\}$, let the likelihood ratios on states given effort a be denoted by $\ell_a^k := \frac{p_{aG}^k}{p_{aB}^k}$. Define:

$$Y^D := \{y^k \in Y : \ell_0^k > \ell_1^k\}.$$
$$Y^U := \{y^k \in Y : \ell_0^k < \ell_1^k\}.$$

Lemma 2. *For any efforts a, a^* with $a < a^*$: $\pi_a^k > \mu_{a^*}^k$ if $y^k \in Y^D$, $\pi_a^k < \mu_{a^*}^k$ if $y^k \in Y^U$ and $\pi_a^k = \mu_{a^*}^k$ if $y^k \in Y \setminus (Y^U \cup Y^D)$.*

Proof. See Appendix A. □

The next lemma is key for our results, since it shows that Y^D is non-empty – there is at least one signal such that the agent is more optimistic than the principal when he shirks. We prove a more general result, that the agent is on average more optimistic than the principal, since it is of independent interest, for conceptual reasons, as well as practically. The proof is of independent economic interest – the result follows from the martingale property of beliefs and Assumption 1. Let $\mathbf{E}_{0,\lambda}(\pi_0^k)$ denote the expectation of the agent's belief, given prior λ and effort 0, and let $\mathbf{E}_{0,\lambda}(\mu_1^k)$ the expectation of the principal's "false" belief, given that the principal believes that the agent is choosing $a = 1$, while in fact he is choosing $a = 0$.[8]

Lemma 3. $\mathbf{E}_{0,\lambda}(\pi_0^k) > \mathbf{E}_{0,\lambda}(\mu_1^k)$, *so that Y^D is non-empty.*

Proof. See Appendix A. □

We have shown that the expectation of the "false belief" held by the principal, μ_1^k, that is induced when the agent performs the experiment $a = 0$, is strictly smaller than the expectation of the true belief π_0^k. Thus there must be some signal realization for which $\pi_0^k > \mu_1^k$. This immediately proves that the agent can increase his continuation value by deviating to low effort.

Lemma 3 follows from Assumption 1. Thus the ratchet effect obtains under a fairly general information structure – most existing work assumes either binary or normal signals. Assumption 1 plays a similar role in Bhaskar and Mailath (2019), which examines in the long run consequences of belief manipulation. Lemmas 2 and 3 are robust to some weakening of Assumption 1 – one can have some signals that violate Assumption 1, as long as the probability of these signals is small. This follows from the fact that the established inequalities in the lemmas are strict, so that they will continue to apply if we have a small perturbation of an information structure that satisfies Assumption 1.

In the light of lemma 2 we may re-write $W(a, a^*)$ as[9]

[8]That is, the principal updates assuming that effort 1 has been chosen, but the distribution on output signals is given by effort 0 and the prior λ.

[9]$W(a^*, a^*) = 0$.

$$W(a,a^*) = \begin{cases} \sum_{y^k \in Y^D} p^k_{a\lambda} V(\pi^k_e, \mu^k_{a^*}) \text{ if } a < a^* \\ \sum_{y^k \in Y^U} p^k_{a\lambda} V(\pi^k_a, \mu^k_{a^*}) \text{ if } a > a^*. \end{cases}$$

The overall payoff of the agent from effort choice a in the first period, given a contract $\boldsymbol{u}$ that seeks to induce effort level a^*, is given by

$$\mathcal{U}(a; a^*, \boldsymbol{u}) = \boldsymbol{u}.\boldsymbol{p}_{a\lambda} - c(a) + \delta W(a, a^*).$$

Definition 3. *Effort a^* is* implementable *in period* 1 *if there exists a spot contract $\boldsymbol{u}$ such that $a = a^*$ maximizes $\mathcal{U}(a; a^*, \boldsymbol{u})$.*

The following theorem and its generalization (Theorem 2 in Section 2.4) are the main negative results of this paper. First, let us define the set of *relevant beliefs*, $\mathcal{M}$, as those beliefs that can arise for some effort choice and some signal realization:

$$\mathcal{M} := \{\mu^k_a, a \in [0,1], k \in \{1, 2.., K\}\}.$$

Theorem 1. *Assume that optimal effort at $t = 2$ is not zero, for all relevant beliefs in $\mathcal{M}$. If $a^* \in (0,1)$, then a^* is **not** implementable at $t = 1$. The extremal efforts* 0 *and* 1 *are implementable.*

Proof. We evaluate the left-hand and right-hand partial derivatives of $W(a, a^*)$ with respect to its first argument at $a = a^* \in (0,1)$, and show that these are inconsistent with the first order conditions for implementing a^*.The left hand partial derivative is given by

$$W_1^-(a^*, a^*) = \sum_{k=1}^{K} (p^k_{1\lambda} - p^k_{0\lambda}) V(\pi^k_{a^*}, \mu^k_{a^*}) + \sum_{y^k \in Y^D} p^k_{a^*\lambda} V^+_\pi(\pi^k_{a^*}, \mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*}. \quad (1)$$

The first term is zero since $V(\mu, \mu)$ is constant, since $\sum_{k=1}^{K}(p^k_{1\lambda} - p^k_{0\lambda}) = 0$. By lemma 1, the right-hand derivative of V is the partial derivative of $\hat{V}$, with respect to its first argument, and thus

$$W_1^-(a^*, a^*) = \sum_{y^k \in Y^D} p^k_{a^*\lambda} \hat{V}_\pi(\pi^k_{a^*}, \mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*}. \quad (2)$$

The partial derivative of $\hat{V}$ equals $\left(\boldsymbol{p}_{\tilde{a}(\mu)G} - \boldsymbol{p}_{\tilde{a}(\mu)B}\right).\boldsymbol{u}_\mu > 0$. From equation 12, $\left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*}$ has the same sign as $\left[p^k_{1G}p^k_{0B} - p^k_{1B}p^k_{0G}\right]$, which is strictly negative if $y^k \in Y^D$. Since each term in the summation is strictly negative, $W_1^-(a^*, a^*) < 0$.The right-hand derivative, $W_1^+(a^*, a^*)$, is bounded below by zero, since $W(a, a^*) \geq 0$ and $W(a^*, a^*) = 0$. If there is uniform optimism, $W_1^+(a^*, a^*) = 0$, since Y^U is empty.

If Y^U is non-empty, then similar arguments as for the left-hand derivative show that $W_1^+(a^*, a^*) > 0$.

The agent's current payoff at $t = 1$ is a smooth function of effort, given the differentiability of the cost of effort and expected utility. Thus the first order conditions for a^* to be optimal for the agent at $t = 1$ are:

$$(\boldsymbol{p}_{1\lambda} - \boldsymbol{p}_{0\lambda}) \, .\boldsymbol{u} - c'(a^*) + \delta W_1^-(a^*, a^*) \geq 0.$$

$$(\boldsymbol{p}_{1\lambda} - \boldsymbol{p}_{0\lambda}) \, .\boldsymbol{u} - c'(a^*) + \delta W_1^+(a^*, a^*) \leq 0.$$

Since $W_1^-(a^*, a^*) < W_1^+(a^*, a^*)$, the two conditions cannot be simultaneously satisfied, thereby proving the main part of theorem.

The extremal efforts, 0 and 1, can be implemented, since one has to only deter deviations in one direction. For implementing $a = 0$, if the signal structure satisfies uniform optimism, then a constant utility schedule, with $u^k = c(0) \forall k$, is the optimal contract. However, if the signal structure does not satisfy uniform optimism, then the agent may have to be punished for higher output levels – he may have an incentive to deviate upwards, since he will be more optimistic than the principal after some output realizations. It suffices to choose utility payments $\boldsymbol{u}$ such that $\boldsymbol{p}_{a\lambda}.\boldsymbol{u} - c(a) + \delta W(a, 0)$ is maximized at $a = 0$. If $c(.)$ is sufficiently convex (to offset the convexity of the W function), then the first order condition suffices:

$$(\boldsymbol{p}_{1\lambda} - \boldsymbol{p}_{0\lambda}) \, .\boldsymbol{u} - c'(0) + \delta W_1^+(0, 0) \leq 0.$$

Similarly, $a = 1$ can also be implemented, by choosing $\boldsymbol{u}$ so that $\boldsymbol{p}_{a\lambda}.\boldsymbol{u} - c(a) + \delta W(a, 1)$ is maximized at $a = 1$. □

Remark 1. *We have assumed that the principal finds it profitable to employ the agent in period* 2 *and to induce non-zero effort after all signal realizations. Neither assumption is required: it suffices that she finds it optimal to employ the agent and induce non-zero effort after* ***some*** *signal in* Y^D*, the set of signals where the agent is more optimistic after downward deviations.*

Remark 2. *This stark result arises from the fact that the agent's participation constraint binds, in equilibrium, in the second period. If the agent were to earn rents, then interior effort may be implementable. Rents can arise if the agent is subject to limited liability. They can also arise due to the agent having some bargaining power or due to the employer paying more than the outside option due to "fairness" considerations.*

The negative result in Theorem 1 is striking: no interior effort level can be implemented in the first period. The ratchet effect implies that the agent can raise his continuation value by shirking a little relative to a^*. To overcome this, incentives today must be high powered, so that a little shirking reduces the agent's current payoff. However, this implies that the agent can also increase his current payoff by overworking relative to a^* – this follows from the fact that current costs and benefits are smooth functions of effort. But over-working cannot *reduce* the agent's continuation value relative to a^*, since the agent can always quit. In other words, the principal can deter downward deviations, but this makes upward deviations profitable. Thus high powered incentives cannot overcome the ratchet effect, contrary to the argument in Milgrom and Roberts (1992). It is also striking that the negative result even though one has contracting before the agent has any private information. Thus ex ante contracting does not help, in contrast with models of hidden information, where contracting at the ex ante stage improves efficiency. The key reason appears to be the lack of inter-temporal commitment.

One question arises: can the principal overcome the impossibility by offering a mechanism with a menu of contracts, rather than a single contract? The answer is no. If a^* is chosen by the agent with probability one, in equilibrium, then the principal assigns probability one to the agent's belief being the same as his, $\mu^k_{a^*}$, after signal y^k. If the agent deviates and has more optimistic beliefs, then he will earn a rent, since he can do so by claiming that he has not deviated. If the agent has more pessimistic beliefs, then he will walk away from the job. This follows from the fact that the principal assigns probability one to the agent's type being $\mu^k_{a^*}$, and therefore, he will not pay rents to this type in order to induce the participation of a more pessimistic type. Of course, if the agent randomizes over effort levels in equilibrium, then the principal faces a non-trivial screening problem, and may find it optimal to pay rents. We address this question in detail in Section 3 in which we construct an equilibrium with the agents exerting random effort in period 1. We use techniques developed in the literature on renegotiation-proof contracts (see Fudenberg and Tirole, 1990), hold-up problem (see Gul, 2001; González, 2004) and dynamic nonlinear pricing (see Bhaskar and Roketskiy, 2021).

2.3 Discussion

We now compare our result with those in models of the ratchet effect arising from ex ante private information – see Laffont and Tirole (1988) and Laffont and Tirole (1993, chapter 10) for a comprehensive discussion. With a continuum of types, their main result is that one cannot have full separation of types, so that there must be some pooling. With binary types, full separation may be possible, but may also be vulnerable to the "take the money and run" strategy, whereby the low type mimics the high type in the first period, and quits in the second period (this is somewhat similar to our finding in Theorem 1). In this case, the equilibrium involves partial separation.

We turn now to a conceptual question: what is the underlying reason for the impossibility result in Theorem 1? This problem arises naturally in agency models where the agent has continuous as well as discrete choices. In our model, the agent had a continuous choice in the first period (effort) as well as a discrete choice in the second period – stay on the job or quit. The agent's overall value function is given by the maximum of two functions: his payoff in both periods when he stays in period 2, and his overall payoff when he quits. Furthermore, each of these payoff functions is locally linear, in the neighborhood of e^*. The maximum of two linear functions is necessarily convex, and as long as the slopes of the linear functions are unequal, it will also have a kink. Thus, the agent's continuation value as a function of effort has a convex kink.[10] Kinks in the maximum value function arise in other contexts, e.g. when the consumer has discrete choices, but they occur only at an isolated set of prices, and are therefore rare. However, in our agency context, the principal *designs* the contract so as to make the worker indifferent between his discrete choices. Thus convex kink in the maximum value function is inevitable, at precisely the point that is relevant.

To our knowledge, the impossibility result we identify is novel, and has not appeared in the recent literature on agency models with experimentation. Bergemann and Hege (1998, 2005), Hörner and Samuelson (2015), Manso (2011) and Kwon (2011) analyze models where agent has a discrete set of actions (usually binary). DeMarzo and Sannikov (2016) and Cisternas (2018) analyze continuous time models, where the agent has continuous action choices. Either of these formulations do not encounter the problems that arise here, since it is the combination of continuous and discrete actions that give rise to the difficulty.

The most closely related antecedent to our impossibility result appears to be the work of Kocherlakota (2004), who examines the design of the optimal unemployment insurance when the agent has two choices to make – effort and private savings. He assumes that the payoff to the agent is linear in effort and concave in savings, and examines whether the principal (who can fully commit) can design a contract that induces an interior level of effort. Since the agent jointly deviate to working less and saving more, the optimal contract must protect against this joint deviation. This implies that the agent's optimization problem is not concave, and thus the first order approach is not valid. More recently, Arie (2016) examines dynamic moral hazard where effort costs exhibit intertemporal dependence, and where the principal can commit. He shows that the first order approach fails in this case. We note that in the absence of commitment, the problem for the principal is significantly worse than the failure of the first-order approach, since sequential rationality precludes her implementing any interior level of effort.

2.4 Generalizing the Impossibility Result

We now show that the impossibility result in Theorem 1 arises under very mild informational conditions. Specifically, it is true generically as long as signals depend upon the state and also depend upon effort. We replace Assumption 1 with the following:

[10]Envelope theorems exist for problems involving discrete choices (see Milgrom and Segal (2002), but they do not deliver differentiability of the maximum value function.

Assumption 1*.

1. *Signals have full support:* $p_{e\omega}^k > 0$ *for all* k, ω *and* $e \in \{0, 1\}$.
2. *There exists* $k \in \{1, 2.., K\}$, *such that* $p_{1\omega}^k \neq p_{0\omega}^k$ *for some* $\omega \in \{G, B\}$ *and* $p_{eG}^k \neq p_{eB}^k$ *for some* $e \in \{0, 1\}$.

As before, we invoke Assumption 2 in to extend the information structure to all effort levels in $[0, 1]$, so that $\forall \omega \in \{G, B\}, p_{e\omega}^k = ep_{1\omega}^k + (1 - e)p_{0\omega}^k$.

In the final period, for any public belief $\mu \in [0, 1]$, every effort $b \in [0, 1]$ is implementable, and the optimal final contract after history can be found using the first-order approach. We assume that effort is productive, so that $\mathbf{E}[y|\omega, e = 1] \geq \mathbf{E}[y|\omega, e = 0]$ for $\omega \in \{G, B\}$, with the inequality being strict for at least one ω. In conjunction with the assumption that the marginal cost of effort is zero at zero, this ensures that for any interior belief, the principal wants to induce positive effort in the static problem. We also assume that the principal always finds it profitable to employ the agent. We now analyze the agent's payoff in the final period, $V(\pi, \mu)$, when his belief is π and differs from the principal's belief μ. Since the agent can always quit, $V(\pi, \mu) \geq 0$. We now show that generically, the agent will get an informational rent, i.e. $V(\pi, \mu) > 0$ either when $\pi > \mu$ or when $\pi < \mu$.

Recall that the informational structure $\mathbf{p} = (p_{0G}, p_{0B}, p_{1G}, p_{1B})$ is an element of $\Delta^{4(K-1)}$ and the vector of possible output realizations $\mathbf{y} = (y_1, .., y_K)$ lies in $\mathbb{R}^K$. We label the pair $(\mathbf{p},\mathbf{y})$ the *parameters of the model.* Thus the parameters of the model live in $\Delta^{4(K-1)} \times \mathbb{R}^K$.

The following lemma is the main step in the proof of the theorem. All proofs for this sub-section can be found in Appendix B.

Lemma 4. *Supppose that informational Assumptions 1* and 2 are satisfied. For almost all parameters of the model,* $V(\pi, \mu) > 0$ *either when* $\pi > \mu$ *or when* $\pi < \mu$.

Proof. See Appendix B. □

We now examine how deviations by the agent from the effort level induced by the principal cause a divergence in belief between the two parties, giving rise to informational rents for the agent. Lemma 2 applies without modification, so that the agent benefits from a downward effort deviation after any signal in Y^D, and from upward deviations after any signal in Y^U. The following lemma shows that at least one of these sets is non-empty.

Lemma 5. *Under Assumption 1*,* $\ell_1^k \neq \ell_0^k$ *for some* $k \in \{1, 2.., K\}$, *so that either* Y^D *or* Y^U *is non-empty.*

Proof. See Appendix B. □

These two lemmas give us the following theorem.

Theorem 2. *Let Assumptions 1* and 2 hold, and let $a^* \in (0,1)$. For almost all parameters of the model,* $(\boldsymbol{p},\boldsymbol{y})$*, effort a^* is* not *implementable at $t = 1$.*

Proof. See Appendix B. □

2.5 Risk neutral agent

Non-implementability also arises if agent is risk neutral as long as long-term commitments are impossible. This also provides a prelude to our analysis of the random effort in section 3, where we assume risk neutrality.

Our key assumption is that contracts are only for one period. This implies that the agent is unable to purchase the project at the outset, possibly due to financial constraints. In the final period, suppose that the common belief is μ. Then the principal can make the agent the residual claimant of the project, by charging a fixed rental, $R(\mu)$. Normalizing the agent's reservation utility to zero, $R(\mu)$ must satisfy the individual rationality constraint:

$$\max_e \left[\boldsymbol{p}_{e\mu}.\boldsymbol{y} - c(e)\right] \geq R(\mu).$$

Since the optimal contract maximizes $R(\mu)$ subject to this constraint, we conclude that $R(\mu)$ equals the left-hand side of the above expression. Thus the principal charges the agent a fee $R(\mu)$. Assume that $\boldsymbol{p}_{1G}.\boldsymbol{y} > \boldsymbol{p}_{1B}.\boldsymbol{y}$ and $\boldsymbol{p}_{0G}.\boldsymbol{y} > \boldsymbol{p}_{0B}.\boldsymbol{y},$ so that the distribution of output conditional on the good state first order stochastic dominates the distribution conditional on a bad state, both when $b = 0$ and when $b = 1$. Thus $R(\mu)$ is increasing in μ. If the agent is offered $R(\mu)$, but has belief $\pi > \mu$, his payoff will be

$$V(\pi,\mu) = \max_b \left[\boldsymbol{p}_{b\pi}.\boldsymbol{y} - c(b)\right] - R(\mu) = R(\pi) - R(\mu).$$

In particular, the derivative is given by

$$V_1^+(\mu,\mu) = \left(\boldsymbol{p}_{\hat{b}(\mu)G} - \boldsymbol{p}_{\hat{b}(\mu)B}\right).\boldsymbol{y} > 0,$$

where $\hat{b}(\mu)$ is optimal effort given μ.

Now consider the first period problem. Suppose that the principal wants to implement effort level a^*. The second period continuation value of the agent when he deviates to $a < a^*$ is given by $W(a, a^*) > 0$. The left hand derivative evaluated at $a = a^*$ is strictly negative, since $V_1^+(\mu,\mu) > 0$. Thus, in order to prevent downward deviations, the agent must be offered more high powered incentives than residual claimancy – his wage payments have to *more variable* than y. However, this implies that the agent has earn more than his outside option today by increasing his effort level beyond a^*, and quitting the job tomorrow, when signals in Y^D are realized. Thus no interior effort level is implementable even when the agent is risk neutral.

This problem can be solved if the agent can sign a long term contract, whereby he commits to buying the project for both periods. The total expected return from this project is

$$\mathcal{R} = \max_{a} \left\{ \sum_{k} p^{k}_{a\lambda} \left\{ y^{k} + \delta \left[\boldsymbol{p}_{\hat{b}(\mu^{k}_{a})\mu^{k}_{a}} . \boldsymbol{y} - c(\hat{b}(\mu^{k}_{a})) \right] \right\} - c(a) \right\}.$$

Thus the agent would be willing to buy this project for this sum. This illustrates that long-term commitment by the parties can solve the ratchet effect. The risk neutral case with equal discount rates also illustrates that only commitment by one side is required. For example, the same outcome can be achieved with only one period commitment by the agent, but with long term commitment by the principal. The agent simply pays the entire amount $\mathcal{R}$ at $t = 1$, and has the option to continue with the project for free in the second period.

The finding that commitment makes a qualitative difference by eliminating the impossibility result is a robust one, and is also true when the agent is risk averse. We do not present this here, since our focus is on the case where commitment is not possible.

The analysis in the paper highlights the importance of long term commitment—in its absence, the ratchet effect makes implementing pure effect impossible. Random effort results in an efficiency loss, as we shall see in the next section. Milgrom and Roberts (1992) emphasize the role of commitment and discuss the Harvard Business School case study of Lincoln Electric. Lincoln Electric used piece rates in order to provide incentives, and had a policy of not revising the piece rate in the light of worker performance. This would provide the worker with rents when the job turned out to be a good one, and so the firm effectively committed not to hold workers to their reservation value in that event. However, if the job turns out to be a bad one, then the piece rate would not meet the worker's reservation utility, unless it was set very generously in the first instance. Thus, a policy of not revising piece rates requires either long term commitments on the worker's part as well, or else the firm to pay the workers rents ex ante.

3 Inducing random effort

Our results so far have shown that the only deterministic effort that the principal can induce in the first period is zero. Our goal in this section is to show, firstly, that the principal can induce positive random effort, i.e. that any first period contract results in an continuation equilibrium where the agent randomizes. Secondly, we show the existence of a solution to the principal's first period optimization problem when she chooses which distribution to induce. Given the complexity of this task, we will make several simplifying assumptions, relative to the previous analysis. We assume that the agent is risk neutral and output is binary: $y \in \{L, H\}$. We will also invoke additional assumptions on the information structure. To reduce notation, we also assume that $\delta = 1$ for both parties.

It may be useful to provide a road-map at this point. Fix an acceptable first period contract (w_H, w_L), with $\Omega := w_H - w_L$ greater than the marginal effort cost at zero, $c'(0)$. We will show that Ω induces a distribution G_Ω over over first period efforts, with support $[\underline{a}_\Omega, \overline{a}_\Omega]$, that is atomless except possibly at $\underline{a}_\Omega$. After first period output is realized, the randomness of first period effort induces two different screening problems, after signals H and L respectively. These screening problems combine moral hazard and adverse selection. The principal's solution to these screening problems give rise to two different continuation values for the agent who chooses first period effort a, $\mathcal{V}_H(a)$ and $\mathcal{V}_L(a)$. These continuation values must be such that $\mathbf{E}[\mathcal{V}_y(a) \mid a]$ plus the first period benefit from a are equalized for all $a-$values in the support $[\underline{a}_\Omega, \overline{a}_\Omega]$. This indifference condition pins down the distribution G_Ω. Finally, we show that the principal's overall payoff is a continuous function of G_Ω, so that an optimum exists. We also provide illustrative numerical examples.

Some preliminaries: because output is binary, we omit the superscripts in the notations for probabilities that we used in the previous section. In particular, we denote the probability of output H conditional on state ω and effort e by $p_{e\omega}$. Let ρ measure the difference in productivity of effort between the good and bad state

$$\rho := (p_{1G} - p_{0G}) - (p_{1B} - p_{0B}).$$

Assumption 1 implies that each of the terms in brackets is positive. We now assume that the difference, $\rho \geq 0$, so that effort is more productive in the good state. Our results also hold for negative ρ as long as its absolute value is small. Let $M(\mu)$ denote the rate at which effort increases probability of high output, given belief μ:

$$M(\mu) := (p_{1B} - p_{0B}) + \mu\rho.$$

We will also assume *uniform optimism* i.e. that the agent's beliefs are decreasing in his effort choice after each signal. This corresponds to an assumption on likelihood ratios, i.e. :

$$\frac{p_{0G}}{p_{0B}} \geq \frac{p_{1G}}{p_{1B}},$$

$$\frac{1-p_{0G}}{1-p_{0B}} \geq \frac{1-p_{1G}}{1-p_{1B}}.$$

In our analysis we rely on the following set of assumptions

Assumption 3.

1. *Effort is more productive in the good state than in the bad state:* $\rho \geq 0$.

2. *Signals satisfy uniform optimism.*

3. The function $c''(x)(p_{0G} - p_{0B} + \rho x)$ is increasing in x.

Note that informational assumptions (points 1 and 2 above) are satisfied if technology is equally productive in both states, so that $\rho = 0$. This is the linear-additive structure that is similar to that in Holmstrom (1999) or Milgrom and Roberts (1992). In this case, the agent is strictly more optimistic after both signals when he shirks. Consequently, our results also hold for ρ positive or negative, as long as it is small in absolute value.

3.1 The final period screening menu

At the end of period one, both parties observe the realization of the output signal, which we will assume to be in $\{L, H\}$. We condition the analysis on this realization without, for notational simplicity, making it explicit. Let the agent's private belief, μ be distributed on the interval $\mathcal{I} = [\underline{\mu}, \overline{\mu}]$, according to the distribution $F(\mu)$. These variations in beliefs arise endogenously, due to the randomness of the agent's first period action. Thus the principal offers a menu $\big(u_L(\mu), \Delta(\mu)\big)_{\mu \in \mathcal{I}}$, where $\Delta(\mu) = u_H(\mu) - u_L(\mu)$. Incentive compatibility trivially implies that if $\Delta(\pi) > \Delta(\mu)$, then $u_L(\pi) < u_L(\mu)$, so that we may identify contracts in the menu by its effort incentive $\Delta(\mu)$ alone.

Suppose that belief-type μ accepts some contract Δ (which may or may not be optimal for this belief type μ). If the effort $b(\mu, \Delta)$, optimal under contract Δ, is neither zero nor one, it satisfies the first order condition[11]

$$M(\mu)\Delta = c'\big(b(\mu, \Delta)\big). \tag{3}$$

Naturally, optimal effort is zero if the incentive in the contract is sufficiently small, i.e., if $M(\mu)\Delta < c'\big(0\big)$. Optimal effort is one if the incentive is large: $M(\mu)\Delta > c'\big(1\big)$.

Note that $b(\mu, \Delta(\mu))$ is the effort of the agent when he chooses an incentive-compatible contract meant for him.[12] Let $V(\mu, \hat{\mu})$ denote the (second-period) payoff of the agent under an incentive compatible menu when agent's belief is μ but he chooses a contract that is designed for an agent with belief $\hat{\mu}$. By the envelope theorem,

$$\frac{d}{d\mu} V(\mu, \mu) = \Delta(\mu)\big[(p_{0G} - p_{0B}) + b(\mu, \Delta(\mu))\rho\big]. \tag{4}$$

Lemma 6. *Suppose that $\rho \geq 0$. A menu $\big(u_L(\mu), \Delta(\mu)\big)_{\mu \in \mathcal{I}}$ is incentive compatible if $\Delta(\mu)$ is increasing and (4) holds.*

[11] Since the rewards are linear in effort while the costs of effort are convex, the second order conditions are satisfied.

[12] The recommended effort, in Myerson's language.

Proof. A menu is globally incentive compatible if and only if for any μ and $\hat{\mu}$,

$$V(\mu,\mu) - V(\mu,\hat{\mu}) \geq 0,$$

where

$$V(\mu,\hat{\mu}) = \Delta(\hat{\mu})[b(\mu,\Delta(\hat{\mu}))M(\mu) + \mu p_{0G} + (1-\mu)p_{0B}] + u_L(\hat{\mu}) - c(b(\mu,\Delta(\hat{\mu})))$$

is the payoff of belief-type μ choosing contract $\Delta(\hat{\mu})$ and choosing effort *optimally.*

Rewrite the incentive compatibility condition as

$$V(\mu,\mu) - V(\hat{\mu},\hat{\mu}) \geq V(\mu,\hat{\mu}) - V(\hat{\mu},\hat{\mu}).$$

Using condition (4) we can rewrite the left hand side of the inequality as

$$V(\mu,\mu) - V(\hat{\mu},\hat{\mu}) = \int_{\hat{\mu}}^{\mu} \Delta(z)(\rho b(z,\Delta(z)) + p_{0G} - p_{0B})dz.$$

The right hand side of the inequality is

$$\begin{aligned} &V(\mu,\hat{\mu}) - V(\hat{\mu},\hat{\mu}) = \\ &\qquad \Delta(\hat{\mu})\left[(\mu-\hat{\mu})(p_{0G}-p_{0B}) + b(\mu,\Delta(\hat{\mu}))M(\mu) - b(\hat{\mu},\Delta(\hat{\mu}))M(\hat{\mu})\right] \\ &\qquad + c(b(\hat{\mu},\Delta(\hat{\mu}))) - c(b(\mu,\Delta(\hat{\mu}))). \end{aligned}$$

Using (3) and integrating by parts, we can replace the difference in costs:

$$\begin{aligned} &c(b(\mu,\Delta(\hat{\mu}))) - c(b(\hat{\mu},\Delta(\hat{\mu}))) = \\ &\int_{\hat{\mu}}^{\mu} c'(b(z,\Delta(\hat{\mu})))b_1(z,\Delta(\hat{\mu}))dz = \\ &\int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})M(z)b_1(z,\Delta(\hat{\mu}))dz = \\ &\Delta(\hat{\mu})[b(\mu,\Delta(\hat{\mu}))M(\mu) - b(\hat{\mu},\Delta(\hat{\mu}))M(\hat{\mu})] - \int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})M(z)b(z,\Delta(\hat{\mu}))\rho dz. \end{aligned}$$

Thus, incentive compatibility can be rewritten as the following set of inequalities:

$$\forall \mu,\hat{\mu}: \int_{\hat{\mu}}^{\mu} \Delta(z)[\rho b(z,\Delta(z)) + p_{0G} - p_{0B}]dz \geq \int_{\hat{\mu}}^{\mu} \Delta(\hat{\mu})[\rho b(z,\Delta(\hat{\mu})) + p_{0G} - p_{0B}]dz. \quad (5)$$

$b(\mu,\Delta)$ is increasing in Δ. Hence if $\rho \geq 0$, the collection of inequalities (5) holds if and only if $\Delta(\mu)$ is increasing in μ. □

Remark 3. *Inequality* (5) *holds strictly when* $\rho = 0$*, and consequently, the lemma still holds for* $\rho < 0$ *but small in absolute value.*

Remark 4. *Chade and Swinkels (2021) analyze the interplay of moral hazard and adverse selection more generally, and provide conditions under which it suffices to consider deviations where type* μ *chooses the recommended effort for type* $\hat{\mu}$ *when choosing* $\Delta(\hat{\mu})$*. Our approach is more direct, since we allow* μ *to choose his optimal effort for contract* $\Delta(\hat{\mu})$*.*

The principal designs the menu that maximizes her second-period expected payoff

$$\max_{\Delta(\cdot)} \int_{\underline{\mu}}^{\overline{\mu}} \left[\boldsymbol{p}_{b(m,\Delta(m)),m} \cdot \boldsymbol{y} - c(b(m,\Delta(m))) - \frac{1-F(m)}{f(m)} \frac{d}{dm} V(m,m) \right] dF(m), \tag{6}$$

under the condition that $\Delta(\cdot)$ is monotone.

Under Assumption 3, the objective is a concave function of b and, therefore, the first order condition is both necessary and sufficient for b (and the corresponding Δ) to be a solution. Our method of constructing an equilibrium is valid only if the solution to the maximization problem (6) satisfies the monotonicity requirement.

Recall that the agent updates his belief on new private and public information that he receives in the first period. Fix the first period output realization to be $y \in \{L, H\}$ and let $\mu_y(a)$ be the agent's second period belief as a function of the first period effort a. These beliefs can be obtained using Bayes rule

$$\mu_H(a) = \frac{\lambda(p_{1G}a + p_{0G}(1-a))}{\lambda(p_{1G}a + p_{0G}(1-a)) + (1-\lambda)(p_{1B}a + p_{0B}(1-a))},$$
$$\mu_L(a) = \frac{\lambda[(1-p_{1G})a + (1-p_{0G})(1-a)]}{\lambda[(1-p_{1G})a + (1-p_{0G})(1-a)] + (1-\lambda)[(1-p_{1B})a + (1-p_{0B})(1-a)]}.$$

We rewrite the principal's second period problem in terms of first period effort rather than agent's beliefs. This allows us to connect this problem across different first period output realizations and, ultimately, formulte the necessary conditon for the mixed strategy equilibrium.

Let $G_\Omega(a)$ be the distribution of the first period effort under the contract Ω. We assume that it has a support $[\underline{a}_\Omega, \overline{a}_\Omega]$ and that it is continuous everywhere, except, perhaps, at $\underline{a}_\Omega$. We also assume that $g_\Omega(a)$ is a density for the continuous part of the distribution. We later verify that all this assumptions hold: this distribution will arise as a solution to a differentiable equation.

Under the uniform optimism, the distribution of the first period effort and the distribution of the second period beliefs conditional on the signal y and the first period contract Ω are connected via the following equations:

$$F_{\Omega,y}\big(\mu_y(a)\big) = 1 - G_\Omega(a),$$
$$f_{\Omega,y}\big(\mu_y(a)\big) = \frac{g_\Omega(a)}{-\mu_y'(a)}.$$

The hazard ratio for the distribution G_Ω is

$$h_\Omega(a) = \frac{G_\Omega(a)}{g_\Omega(a)}.$$

Let $M_y(a) = M(\mu_y(a))$.

We rewrite the virtual surplus—the expression under the integral in (6)—as a function of the first period effort and replace the hazard ratio with a parameter h and define $\tilde{b}_y(a,h)$ to be the value of b that maximizes it:

$$\tilde{b}_y(a,h) = \arg\max_{b\in[0,1]} \left\{ \boldsymbol{p}_{b\mu_y(a)} \cdot \boldsymbol{y} - c(b) + h\mu_y'(a)(\rho b + p_{0G} - p_{0B}) \frac{c'(b)}{M_y(a)} \right\},$$

Since the objective is concave, $\tilde{b}$ is an interior maximizer for a given a and h if and only if it solves

$$y_h[M_y(a)]^2 = c'(\tilde{b})M_y(a) - \mu_y'(a)h\Big[(c'(\tilde{b}) + \tilde{b}c''(\tilde{b}))\rho + c''(\tilde{b})(p_{0G} - p_{0B})\Big]. \qquad (7)$$

There exists a unique differentiable solution to this equation because the right-hand side is continuous and strictly increasing in $\tilde{b}$. Let $\beta_y(a,h)$ be the solution. If $\beta_y(a,h)$ is negative for some values of a and h, then it means that the incentives to exert effort are insufficient and the optimal effort for those values is zero. To account for it, let

$$\tilde{b}_y(a,h) = \max\{0, \beta_y(a,h)\}.$$

We define a threshold for the hazard ratio $\tilde{h}_y(a)$ such that if the hazard ratio for a given a is above this threshold, the effort induced following the signal y is zero:

$$\tilde{h}_y(a) := \frac{y_h[M_y(a)]^2 - c'(0)M_y(a)}{-\mu_y'(a)\Big[c'(0)\rho + c''(0)(p_{0G} - p_{0B})\Big]}.$$

The threshold $\tilde{h}_y(a)$ plays an important role: it determines the conditions under which the agent is not incentivized by the principal—this occurs when the agent is sufficiently pessimistic about the state of the world.

The effort that maximizes principal's payoff is monotone in both the hazard ratio h and the past effort a. This property is useful in testing whether one could find and incentive compatible menu that would induce such an effort.

Lemma 7. *Under Assumption 3, function $\tilde{b}_y(a,h)$ is decreasing in both arguments: $\frac{\partial \tilde{b}_y(a,h)}{\partial a} \leq 0$ and $\frac{\partial \tilde{b}_y(a,h)}{\partial h} \leq 0$.*

Proof. See Appendix C. □

We use the function $\tilde{b}_y(a,h)$ to construct the induced effort (as a function of belief) and then to derive the menu of optimal contracts. Let $\Delta_{\Omega,y}(\mu)$ be a solution to equation (3), namely

$$\Delta_{\Omega,y}(\mu) := \frac{c'\left(\tilde{b}_y\left(\mu_y^{-1}(\mu), h_\Omega\left(\mu_y^{-1}(\mu)\right)\right)\right)}{M(\mu)}.$$

The following lemma presents a simple test for the monotonicity of incentives which is required for incentive compatibility of the menu in period 2.

Lemma 8. *If $h_\Omega(a)$ is increasing in a and Assumption 3 holds, then $\Delta_{\Omega,y}(\mu)$ is increasing in μ.*

Proof. See Appendix C. □

Let us sum up the results of this section so far: we have constructed the optimal second period screening menu, and the agent's best response to this menu as a function of the distribution of the first period effort (or, more precisely, of its hazard ratio) and the first period contract. As the next step, we derive conditions that pin down the distribution of the first period effort. The agent's second period payoff depends on his private information after each output realization, which in turn is determined by the distribution of his first period effort. We look for the distribution with the following property: when the agent increases his first period effort, the resulting increase in his first period payoff must be exactly offset by a reduction in the expected information rent in the second period.

3.2 The first period effort

To find the hazard ratio $h_\Omega(a)$, we compute the expected payoff that the agent receives by choosing effort a. Observe that when the agent chooses the highest equilibrium effort level $\bar{a}$, he is most the most pessimistic type after both signal realizations, due to our assumption of uniform. Thus his second period continuation value is zero. The necessary condition for the optimality of mixed strategy is that the payoff of the agent has to be the same for any effort $a \in [\underline{a}_\Omega, \overline{a}_\Omega]$:

$$\begin{aligned} \Omega[\lambda(p_{1G} - p_{0G}) + (1-\lambda)(p_{1B} - p_{0B})](\overline{a}_\Omega - a) - [c(\overline{a}_\Omega) - c(a)] \\ = \Pr(H|a)\mathcal{V}_H(a) + \Pr(L|a)\mathcal{V}_L(a), \end{aligned} \tag{8}$$

where $\mathcal{V}_y(a) := V_y(\mu_y(a), \mu_y(a)),\ y \in \{L, H\}$ is the second period payoff of the agent who exerted effort a and produced output y in period 1. The value functions on the right-hand side of the equation depend on the distribution of effort through the hazard ratio $h_\Omega(a)$. In particular,

$$\mathcal{V}_y(a) = \int_a^{\overline{a}_\Omega} \frac{-\mu_y'(z)c'(\tilde{b}_y(z, h_\Omega(z)))}{M_y(z)}[p_{0G} - p_{0B} + \rho\tilde{b}_y(z, h_\Omega(z))]dz$$

Also, recall that

$$\Pr(H \mid a) = \lambda(ap_{1G} + (1-a)p_{0G}) + (1-\lambda)(ap_{1B} + (1-a)p_{0B}).$$

Lemma 9. *Equation (8) has a unique solution $h_\Omega(a)$. Moreover, the solution is jointly continuous in (a, Ω).*

Proof. Equation (8) is a Volterra equation of the first kind in the Urysohn form with a degenerate linear kernel (Polyanin and Manzhirov, 2008, p. 676). Let

$$v_y(a,b) = \frac{\mu_y'(a)}{M_y(a)} c'(b)(p_{0G} - p_{0B} + b\rho), \; y \in \{L, H\}.$$

The partial derivatives of this function are

$$\begin{aligned}
v_{y1}(a,b) &= \frac{\partial}{\partial a} v_y(a,b) = \frac{\mu_y''(a) M_y(a) - \rho[\mu'(a)]^2}{[M_y(a)]^2} c'(b)(p_{0G} - p_{0B} + b\rho), \\
v_{y2}(a,b) &= \frac{\partial}{\partial b} v_y(a,b) = \frac{\mu_y'(a)}{M_y(a)} (p_{0G} - p_{0B} + b\rho) c''(b) + \rho c'(b) \le 0.
\end{aligned}$$

Also let $\mathcal{P}(a) = \Pr\{H \mid a\}$ and $\mathcal{P}'(a) = \lambda\rho + p_{1B} - p_{0B}$.

By differentiating equation (8) twice we rewrite it as a first order ODE:

$$\begin{aligned}
c'' = v_{L1} &+ v_{L2}(\tilde{b}_{L1} + \tilde{b}_{L2} h') + 2\mathcal{P}'(v_H - v_L) \\
&+ \mathcal{P}(v_{H1} - v_{L1} + v_{H2}(\tilde{b}_{H1} + \tilde{b}_{H2} h') \\
&- v_{L2}(\tilde{b}_{L1} + \tilde{b}_{L2} h')).
\end{aligned}$$

Equivalently,

$$h' = \frac{c'' - \Big[2\mathcal{P}'(v_H - v_L) + \mathcal{P}(v_{H1} + v_{H2}\tilde{b}_{H1}) + (1-\mathcal{P})(v_{L1} + v_{L2}\tilde{b}_{L1})\Big]}{\mathcal{P} v_{H2}\tilde{b}_{H2} + (1-\mathcal{P}) v_{L2}\tilde{b}_{L2}}. \tag{9}$$

Recall that $\tilde{b}_y$ is not differentiable when $h = \tilde{h}$ (consequently, h is not differentiable at that point too). To work around this issue, we rewrite the equation (9) as two and use the left derivative of $\tilde{b}$ which always exists and equal to the derivative of $\beta(a, h)$:

$$\begin{aligned}
h_1' &= \frac{c'' - [2\mathcal{P}' v_H + \mathcal{P}(v_{H1} + v_{H2}\beta_{H1})]}{\mathcal{P} v_{H2}\beta_{H2}}, \; \tilde{h}_L \le h_1 \le \tilde{h}_H, \qquad (10) \\
h_2' &= \frac{c'' - [2\mathcal{P}'(v_H - v_L) + \mathcal{P}(v_{H1} + v_{H2}\beta_{H1}) + (1-\mathcal{P})(v_{L1} + v_{L2}\beta_{L1})]}{\mathcal{P} v_{H2}\beta_{H2} + (1-\mathcal{P}) v_{L2}\beta_{L2}}, \; 0 \le h_2 \le \tilde{h}_L.
\end{aligned}$$

The right-hand sides for both equations are continuous in a and Lipschitz-continuous in h, therefore both equation have unique sollutions given their respective boundary conditions. These boundary conditions are

$$h_1(\overline{a}_\Omega) = \tilde{h}_H(\overline{a}_\Omega),$$
$$h_2(\tilde{a}_\Omega) = h_1(\tilde{a}_\Omega) = \tilde{h}_L(\tilde{a}_\Omega).$$

To find the lowest level of effort possible in equilibrium we examine $h_2(a)$: if $h_2(a) > 0$ for any $a \geq 0$, then $\underline{a}_\Omega = 0$. Otherwise, $\underline{a}_\Omega$ solves $h_2(\underline{a}_\Omega) = 0$. In the former case, the distribution of efforts has a mass at the lower bound of the support, whereas in the latter case, the distribution is continuous everywhere.

Finally, we obtain the solution for equation (8) by gluing the functions h_1 and h_2 together:

$$h_\Omega(a) = \begin{cases} h_1(a), & \text{if } a \in [\tilde{a}_\Omega, \overline{a}_\Omega], \\ h_2(a), & \text{if } a \in (\underline{a}_\Omega, \tilde{a}_\Omega), \\ 0, & \text{if } a = \underline{a}_\Omega. \end{cases}$$

Note that h is continuous everywhere except, perhaps, at $\underline{a}_\Omega$. It is also differentiable everywhere except at $\tilde{a}_\Omega$ and $\underline{a}_\Omega$.

Interestingly, Ω does not enter the differential equations, which means that the solution depends on Ω only through the boundary condition. This implies that the functions h_1 and h_2 are continuous in (a, Ω) jointly and so is $h(a)$ except perhaps at $a = \underline{a}_\Omega$.

$\square$

Given the solution h_Ω of equation (8), we can find the distribution of the first period efforts. Depending on the magnitude of Ω, there are two cases. If $h_\Omega(a)$ is discontinuous at $\underline{a}_\Omega$ (which is the case when Ω is small and $\underline{a}_\Omega = 0$), then the distribution of efforts has a mass point at 0 and continuous everywhere else. Otherwise the distribution of efforts is continuous. In either case:

$$G_\Omega(a) = \begin{cases} 0, & \text{if } a < \underline{a}_\Omega, \\ \lim\limits_{a \to \underline{a}_\Omega + 0} e^{-\int\limits_a^{\overline{a}_\Omega} \frac{1}{h_\Omega(x)} dx}, & \text{if } a = \underline{a}_\Omega, \\ e^{-\int\limits_a^{\overline{a}_\Omega} \frac{1}{h_\Omega(x)} dx}, & \text{if } a \in (\underline{a}_\Omega, \overline{a}_\Omega), \\ 1 & \text{if } , a \geq \overline{a}_\Omega. \end{cases}$$

Example 1. *We illustrate the second period contract with an example. Consider a setting with quadratic cost of effort $c(x) = \frac{x^2}{2}$. Let us set the parameters of the model to*

$$p_{1G} = 0.9, \quad p_{0G} = 0.6,$$
$$p_{1B} = 0.2, \quad p_{0B} = 0.1,$$
$$\lambda = 0.5, \quad y_h = 2.$$

We solve equation (7) *for the second-period effort:*

$$\tilde{b}_y(a,h) = \max\left\{0, \frac{y_h[M_y(a)]^2 + \mu_y'(a)(p_{0G} - p_{0B})h}{M_y(a) - 2\rho h \mu_y'(a)}\right\}$$

Note that under Assumption 3, the denominator in the above expression is positive and, therefore, the induced effort is zero if and only if

$$h \geq \tilde{h}_y(a) = \frac{y_h[M_y(a)]^2}{-\mu_y'(a)(p_{0G} - p_{0B})}.$$

This condition sets the three regions for the hazard ratio for which the induced effort is (see Figure 1):

(a) zero after both H and L signals;

(b) zero only following signal L;

(c) positive following both H and L signals.

This inequality is also a boundary condition for the the Volterra equation (8) *which we solve using the quadrature method. Indeed, at the upper bound of the support $\overline{a}_\Omega$, the induced effort is zero following both signals, therefore $h_\Omega(\overline{a}_\Omega) = \tilde{h}_H(\overline{a}_\Omega)$.*

To illustrate the optimal menu of contracts in second period, suppose that the principal sets the first period incentives to $\Omega = 1$. Given the first period contract, the support of the effort distribution is $[0.0727, 0.2]$. The resulting hazard ratio $h_\Omega(a)$, distribution of first period efforts $G_\Omega(a)$, induced second-period efforts $\tilde{b}_H(a, h_\Omega(a))$ and $\tilde{b}_L(a, h_\Omega(a))$, and second period incentives $\Delta_{\Omega H}(\mu_H(a))$ and $\Delta_{\Omega L}(\mu_L(a))$ are depicted on Figure 1.

Note that the support of the first period effort is partitioned into two regions:

- *(i) for sufficiently high effort a, the principal induces positive effort in period 2 only after signal H;*

- *(ii) for low effort a, the principal provides enough incentives for effort to be positive regardless of the signal, but the effort induced after signal H is the higher of the two.*

This pattern of efforts can be explained using two important features of the model: on one hand, the agent should receive rent in the second period that decreases to zero when her first period effort approaches $\overline{a}_\Omega$ from below – the promised rent in period 2 is part of the incentives to exert high effort in the first period. On the other hand, effort is more productive after signal H. Therefore, if for some values of a, induced effort $\tilde{b}_H$ is close to zero, the induced effort $\tilde{b}_L$ must be zero. After all, a high first period effort together with signal L indicates that the bad state of the world is likely.

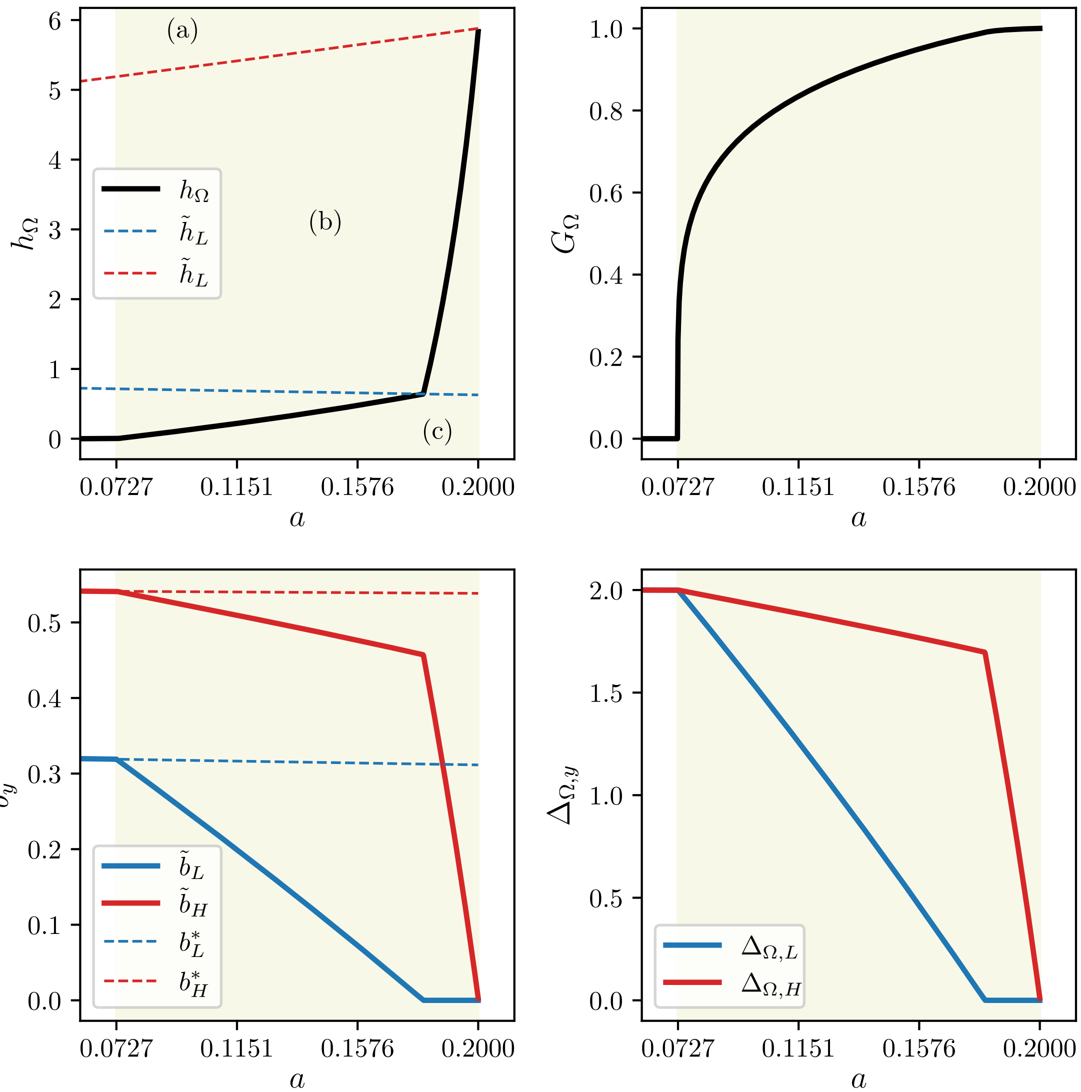

Figure 1: Equilibrium second-period contract and agent's effort choice conditional on the first period contract $\Omega = 1$.

Note: Top left is the hazard ratio h_Ω (dashed lines indicated the borders of the three regions; see equation (10)); Top right is the distribution of first period efforts G_Ω; Bottom left is the induced second-period efforts $\tilde{b}_H$ and $\tilde{b}_L$ (dashed lines indicate conditional first-best effort levels); Bottom right indicate second-period incentives $\Delta_{\Omega,H}$ and $\Delta_{\Omega,L}$; shaded region indicates the support of G_Ω; blue lines indicate choices after output L and red lines indicate choices after output H.

3.3 The first period contract

In the previous section we have found the optimal second-period contract as a function of first period incentives Ω. In this final step, we set up the principal's maximization problem that pins down the optimal first period contract.

Let $B_y(a,\Omega) := \tilde{b}_y(a, h_\Omega(a))$. The principal's total payoff is the sum of his payoff over the two periods. The second-period payoff that we found in the previous section is

$$\pi_{2y}(a,\Omega) = -\mu_y'(a)\left[\boldsymbol{p}_{B_y(a,\Omega),\mu_y(a)} \cdot \boldsymbol{y} - c(B_y(a,\Omega)) + \mu_y'(a)h_\Omega(a)\frac{d}{d\mu}V_y(\mu,\mu)\Big|_{\mu=\mu_y(a)}\right]$$

$$\pi_2(a,\Omega) = \Pr(H|a)\pi_{2H}(a,\Omega) + \Pr(L|a)\pi_{2L}(a,\Omega)$$

and the payoff that the principal receives in the first period. The later is equal to the entire surplus generated in the first period:

$$\pi_1(a) = \boldsymbol{p}_{a\lambda} \cdot \boldsymbol{y} - c(a).$$

Therefore, the total expected payoff is

$$\pi(\Omega) = G_\Omega(\underline{a}_\Omega)[\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega,\Omega)] + \int_0^1 [\pi_1(a) + \pi_2(a,\Omega)]g_\Omega(a)da.$$

Note that every item on the right-hand side of the equation depends on Ω.

The function $\pi(\Omega)$ is flat for all $\Omega \geq \Omega_m$, where Ω_m is (a finite) level of incentives that induces $a = 1$ with certainty. This arises due to the assumed risk-neutrality of the agent. Therefore when maximizing this function, we limit our attention to the compact set $[0, \Omega_m]$.

Lemma 10. *$\pi(\Omega)$ is continuous on $[0,\Omega_m]$, and therefore, there exists $\Omega_e \in [0,\Omega_m]$ that maximizes $\pi(\Omega)$.*

Proof. See Appendix C. □

The argument in Lemma 10 completes our construction of equilibrium. If the equilibrium exists, the equilibrium strategies are as follows:

(i) For every Ω, first period effort $a \sim G_\Omega(a)$.

(ii) For each output realization y, the second period contract $\Delta_{\Omega,y}(m)$ that solves equation (3) when the effort b is replaced with the function $\tilde{b}_y(\mu_y^{-1}(m), h_\Omega(\mu_y^{-1}(m)))$.

(iii) Second period effort $b(\mu_y(a), \Delta)$ that solves equation (3).

(iv) Finally, the first period contract Ω_e that maximizes $\pi(\Omega)$.

We summarize the results of Section 3 in the following theorem.

Theorem 3. *If the unique solution of equation* (8) $h_\Omega(a)$ *is increasing in the first period effort* a *for every first period level of incentives* Ω*, an equilibrium exists for each* Ω*, and there exists a profit maximizing first period contract,* Ω_e*.*

The condition in Theorem 3 is a sufficient condition for monotonicity of the second period incentives. As we show in Lemma 6, monotone incentives guarantee that the second period menu is incentive compatible. As an alternative, instead of examining function h_Ω, one could directly check that the resulting menus $\Delta_{\Omega,y}$ are monotone in agent's private belief. In simpler models of screening with endogenous types (e.g., Fudenberg and Tirole, 1990; Gul, 2001; González, 2004; Bhaskar and Roketskiy, 2021), this condition can be replaced with an assumption on the concavity of surplus. However, because our model has two different screening problems following two output realizations in period 1, finding a tractable condition imposed directly on the surplus function is difficult.

Example 1 (cont.). *We continue the example from the previous section. Using the same set of parameters, we illustrate how the principal's overall payoff depends on the first period incentives. As shown on Figure 1, the principals payoff is maximized at the intermediate level of the first period incentives* $\Omega_e = 2.61$*. When the optimal contract is offered in equilibrium, the first period effort is distributed on* $[0.38; 0.52]$*.*

Consider a modified example, in which the gain from a successful project is much higher at $y_h = 5$*, and the rest of the parameters are the same. In this case, the principal finds it optimal to induce the maximal possible effort* $a = 1$ *in the first period. As shown on on Figure 1, the principal achieves this with a contract* $\Omega_e = 7.35$ *(or higher). The agent responds to this contract by exerting* $a = 1$ *with certainty. In the second period, the principal induces the conditional first-best efforts by offering the contract* $\Delta_H = \Delta_L = y_h$ *and extracts the full rent from the agent.* [13]

Note that in both cases, the optimal incentives $\Omega_e > y_h$*. Although, it is true in general, the intuition for this is especially clear in the second case. If the principal were to offer* $\Omega = y_h$*, the agent would exploit the contract by shirking in the first period and, as a result, would become more optimistic than the principal when the public signal is realized. Since the second period contract is determined by the principal's equilibrium beliefs about the agent's productivity, the agent would benefit from exerting an effort that is higher than expected. To make sure that such behavior is not optimal, the principal must provide more high-powered incentives in period 1.*

The difference between the first and the second cases of this example is subtle: in the second case, the principal ensures that such a behavior does not emerge; in

[13] In this example it is never optimal for the principal to offer $\Omega_e = 0$, due to the fact that the marginal cost of effort at zero is zero. A straightforward way to modify the example would be to set $c(x) = kx + \frac{x^2}{2}, k > 0$. When k is large enough, it is optimal for the principal to induce no effort in equilibrium.

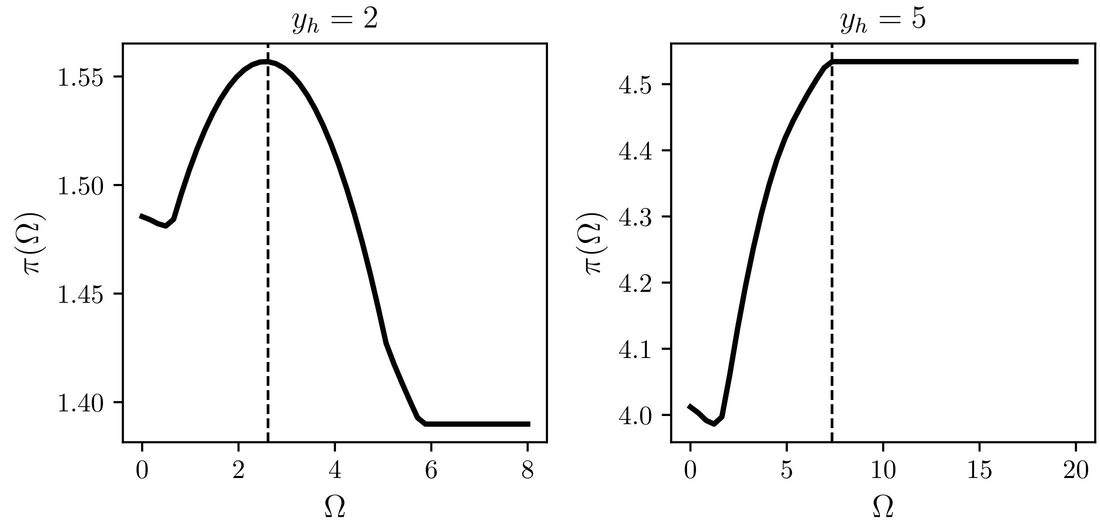

Figure 2: The principal's payoff conditional on the first period contract Ω.
Note: Left is the payoff when $y_h = 2$ (see Example 1); Right is the payoff when $y_h = 5$ and the rest of the parameters are the same as in Example 1.

the first case, the principal ensures that such a behavior does not emerge more often than indicated by the equilibrium distribution $G_{\Omega_e}(a)$. In the latter case the principal allows for some degree of asymmetric information. However, in contrast with our impossibility result in Theorems 1 and 2, the assymetry of information is anticipated and correctly quantified as the principal j expects the agent to exerts an effort below $\overline{a}_{\Omega_e}$ in order to collect rent in period 2.

3.4 The inefficiency of random effort

Suppose that the principal's optimal contract Ω_e induces a mixed strategy equilibrium in the continuation game, with support $[\underline{a}_\Omega, \overline{a}_\Omega]$. This has the following efficiency implications. First period effort is necessarily inefficient, since the efficient effort level would normally be unique.[14] Second period effort is also distorted downwards for every type $a < \overline{a}_\Omega$. However, when the principal induces extremal effort, the agent has no private information in the second period, and consequently, second-period output is undistorted. Therefore, the ratchet effect could lead the principal to induce extremal efforts, even when this is very inefficient in terms of the first-period payoffs. In other words, the ratchet effect can lead the principal to favor either maximal possible effort or no effort at all. If we assume that the cost of effort becomes

[14]It is also true that the agent who chooses $a < \overline{a}_\Omega$ is subject to additional risk, but this has no efficiency implications due to his assumed risk-neutrality.

prohibitive at very high effort levels, then over-provision of effort is unlikely.

4 Conclusion

We have studied classical moral hazard where the agent is not subject to limited liability. Our novel ingredients are learning and the no long-term commitment: both parties are learning about the efficacy of a new technology, and neither can make long term commitments. We find that no pure effort level can be implemented in the first period, and consequently, the optimal contract involves random effort. This has negative efficiency consequences, and may deter the introduction of new technology. Our negative results, on the non-implementability of deterministic effort, are quite general. However, the analysis of random effort is quite complex since we have the interaction of moral hazard and adverse selection, where the agent's private information is endogenously generated. This combination leads to complications that do not arise in other contexts with randomization, such as the hold-up problem or moral hazard with renegotiation. We therefore make several simplifying assumptions, including agent risk neutrality and uniform optimism, which we would like to relax in future work.

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Appendix A: Proofs related to Theorem 1

Proof of Lemma 1

From claim 1, the optimal contract in the final period $\boldsymbol{u}$ must satisfy the first order conditions for $\hat{b}$ to be optimal at μ: [15]

$$\boldsymbol{u}.(\boldsymbol{p}_{1\mu} - \boldsymbol{p}_{0\mu}) = c'(\hat{b}) > 0. \tag{11}$$

In an optimal static contract, utility payments u^k must be increasing, since they are ordered in terms of the likelihood ratio. Thus $\boldsymbol{u}$ can be written as $\boldsymbol{u} = z.\mathbf{1}+\tilde{\boldsymbol{u}}$, where $z.\mathbf{1}$ is a vector where each component equals z, and $\tilde{u}^k > 0$ if $y^k \in Y^H$, and if $\tilde{u}^k < 0$ if $y^k \in Y^L$.The agent's payoff from his optimal effort choice at π is no less than his payoff from choosing $\hat{b}$ at π, which equals

$$\left[\boldsymbol{u}.\boldsymbol{p}_{\hat{b}\pi} - c(\hat{b})\right] - \left[\boldsymbol{u}.\boldsymbol{p}_{\hat{b}\mu} - c(\hat{b})\right] = (\pi-\mu)\boldsymbol{u}.(\boldsymbol{p}_{\hat{b}G} - \boldsymbol{p}_{\hat{b}B}) = (\pi-\mu)\tilde{\boldsymbol{u}}.(\boldsymbol{p}_{\hat{b}G} - \boldsymbol{p}_{\hat{b}B}),$$

since $\mathbf{1}.(\boldsymbol{p}_{\hat{b}G} - \boldsymbol{p}_{\hat{b}B}) = 0$. Assumption 1 implies that $p^k_{\hat{b}G} - p^k_{\hat{b}B} > 0$ if $y^k \in Y^H$ and $p^k_{\hat{b}G} - p^k_{\hat{b}B} < 0$ if $y^k \in Y^L$, and thus $\tilde{\boldsymbol{u}}.(\boldsymbol{p}_{\hat{b}G} - \boldsymbol{p}_{\hat{b}B}) > 0$.

Letting $\tilde{b}(\pi)$ denotes the optimal effort choice at belief π,

$$\hat{V}(\pi,\mu) = \boldsymbol{p}_{\tilde{b}(\pi)\pi}.\boldsymbol{u} - c(\tilde{b}(\pi)).$$

The derivative with respect to π, evaluated at (π,μ), equals

$$V_1(\pi,\mu) = \left(\boldsymbol{p}_{\tilde{b}(\pi)G} - \boldsymbol{p}_{\tilde{b}(\pi)B}\right).\boldsymbol{u} + \frac{d\tilde{b}}{d\pi}\left[\left(\boldsymbol{p}_{1\pi} - \boldsymbol{p}_{0\pi}\right).\boldsymbol{u} - c'(\tilde{b}(\pi))\right] = \left(\boldsymbol{p}_{\tilde{b}(\pi)G} - \boldsymbol{p}_{\tilde{b}(\pi)B}\right).\boldsymbol{u},$$

since the second term is zero by the envelope theorem.

Given any $\pi > \mu, \hat{V}(\pi,\mu)$ is the supremum of linear functions, i.e.

$$\hat{V}(\pi,\mu) = \sup_b\{(\pi-\mu)\boldsymbol{u}.(\boldsymbol{p}_{bG} - \boldsymbol{p}_{bB})\},$$

and is thus convex in π. Since V equals the maximum of $\hat{V}$ and 0, it is also convex in π.

Proof of Lemma 2

Write the difference in beliefs as

$$\begin{aligned}\pi^k_b - \mu^k_{b^*} &= \frac{\lambda p^k_{bG}}{p^k_{b\lambda}} - \frac{\lambda p^k_{b^*G}}{p^k_{b^*\lambda}} \\ &= \frac{\lambda}{p^k_{b\lambda}p^k_{b^*\lambda}}\left(p^k_{bG}p^k_{b^*\lambda} - p^k_{b^*G}p^k_{b\lambda}\right).\end{aligned}$$

[15]If $\hat{b}(\mu) = 1$, then equation (11) applies to the left hand derivative of $c(b)$ at 1.

Using the fact that p^k_{bG} is a convex combination of p^k_{1G} and p^k_{0G} (and similarly $p^k_{b^*G}, p^k_{b^*\lambda}$ and $p^k_{b\lambda}$), this can be re-written as

$$\pi^k_b - \mu^k_{b^*} = \frac{\lambda(1-\lambda)}{p^k_{b\lambda} p^k_{b^*\lambda}}(b^* - b)\left[p^k_{0G}p^k_{1B} - p^k_{0B}p^k_{1G}\right]. \tag{12}$$

For any $b < b^*$,the sign of $\pi^k_b - \mu^k_{b^*}$ depends only on the sign of $p^k_{0G}p^k_{1B} - p^k_{0B}p^k_{1G}$, i.e. on the sign of $\ell^k_0 - \ell^k_1$, thereby proving the lemma.

Proof of Lemma 3

et $\mathbf{E}_{1,\lambda}(\mu^k_1)$ denote the expectation of the belief of the principal when she correctly conjectures that the agent is choosing $e = 1$. From the martingale property of beliefs, $\mathbf{E}_{0,\lambda}(\pi^k_0) = \mathbf{E}_{1,\lambda}(\mu^k_1) = \lambda$, i.e.

$$\sum_{k=1}^{K} p^k_{0\lambda}\pi^k_0 = \sum_{k=1}^{K} p^k_{1\lambda}\mu^k_1.$$

Subtract $\sum_{k=1}^{K} p^k_{0\lambda}\mu^k_1$ from both sides to get

$$\sum_{k=1}^{K} p^k_{0\lambda}(\pi^k_0 - \mu^k_1) = \sum_{k=1}^{K}(p^k_{1\lambda} - p^k_{0\lambda})\mu^k_1.$$

Observe that $\lambda \sum_Y (p^k_{1\mu} - p^k_{0\mu}) = 0$, since the sum of the difference between two probability distributions is zero. Consequently,

$$\sum_{k=1}^{K} p^k_{0\lambda}(\pi^k_0 - \mu^k_1) = \sum_{k=1}^{K}(p^k_{1\lambda} - p^k_{0\lambda})(\mu^k_1 - \lambda).$$

Under Assumption 1, for any k, $(p^k_{1\lambda} - p^k_{0\lambda})$ has the same sign as $(\mu^k_1 - \lambda)$ – i.e. a signal that has higher probability under high effort is also informative of the job being easier. Since there is some informative signal, we conclude that $\sum_{k=1}^{K} p^k_{0\lambda}(\pi^k_0 - \mu^k_1) > 0$, i.e. the expectation of the difference in beliefs under the experiment $e = 0$ is strictly positive. Thus there must be some signal y^k such that $\pi^k_0 > \mu^k_1$.

Appendix B: Proofs related to Theorem 2

Proof of Lemma 4

Let $\tilde{V}(\pi, \mu)$ denote the agent's payoff when he accepts the job and chooses b_μ:

$$\tilde{V}(\pi, \mu) = (\pi - \mu)(p_{b_\mu G} - p_{b_\mu B}).u_\mu. \tag{13}$$

To prove the lemma, it clearly suffices to show that $\tilde{V}(\pi, \mu)$ is non zero when $\pi \neq \mu$. That is, we need to show that generically,

$$(p_{bG} - p_{bB}).u_\mu = (1-b)(p_{0G} - p_{0B}).u_\mu + b(p_{1G} - p_{1B}).u_\mu \tag{14}$$

is not equal to zero at $b = b_\mu$. Since (14) is an affine function of b, it equals zero at most one value of e or at every value of b.

If (14) is zero at every value of b, this implies $(p_{0G} - p_{0B}).u_\mu = 0$ and $(p_{1G} - p_{1B}).u_\mu = 0$. We now show that this violates Assumption 1* when output signals are binary, and will not be satisfied generically when $K > 2$. Observe that since $b_\mu > 0$, u_μ is not a constant vector.

Let p and q be arbitrary probability distributions on Y, so that $p, q \in \Delta^{K-1}$, the $K-1$ dimensional simplex. Given a vector $u \in \mathbb{R}^K$, let $\tilde{u}$ denote the $K-1$ dimensional vector $(u^k - u^1)_{k=2}^K$. Let $\tilde{p} = (p^k)_{k=2}^K$ and $\tilde{q} = (q^k)_{k=2}^K$. It is easy to verify that $(p-q).u = (\tilde{p} - \tilde{q})\tilde{u}$, which allows us to write

$$(p_{0G} - p_{0B}).u_\mu = (\tilde{p}_{0G} - \tilde{p}_{0B}).\tilde{u}_\mu. \tag{15}$$

$$(p_{1G} - p_{1B}).u_\mu = (\tilde{p}_{1G} - \tilde{p}_{1B}).\tilde{u}_\mu. \tag{16}$$

Consider first the case of binary signals, i.e. $K = 2$. In this case, each of the above probability vectors (e.g. $\tilde{p}_{0G}$) are one-dimensional scalars, and $\tilde{u}_\mu$ is a non-zero real number, since it must provide incentives for positive effort. Assumption 1* then ensures that both 15 and 16 cannot be zero.

Now consider $K > 2$. Since $\tilde{u}_\mu$ is not equal to the null vector, the set of $(\tilde{p}_{0G} - \tilde{p}_{0B})$ values such that $(\tilde{p}_{0G} - \tilde{p}_{0B}).\tilde{u}_\mu = 0$ defines a hyperplane in $\mathbb{R}^{K-1}$, that is of Lebesgue measure zero in $\mathbb{R}^{K-1}$. Similarly, the set of values of $(\tilde{p}_{1G} - \tilde{p}_{1B})$ such that $(\tilde{p}_{1G} - \tilde{p}_{1B}).\tilde{u}_\mu = 0$ also lies in the same hyperplane. Since the set of distributions $(\tilde{p}_{1G}, \tilde{p}_{1B}, \tilde{p}_{0G}, \tilde{p}_{0B})$ such that $(\tilde{p}_{1G} - \tilde{p}_{1B})$ and $(\tilde{p}_{0G} - \tilde{p}_{0B})$ lie in the same hyperplane is of Lebesgue measure zero in $\mathbb{R}^{K-1}$, we conclude that the both (15) and (16) cannot be zero generically. This establishes that for generic information structures, (14) cannot be zero at every value of b.

If there is no value of b such that $(p_{bG} - p_{bB}).u_\mu = 0$, the lemma is proved. So let $\breve{b}$ denote the single value of b such that $(p_{eG} - p_{e_B}).u_\mu = 0$. Let $\varphi(b, \mu)$ denote the expected wage cost to the principal of inducing effort b at belief μ.We now show that for generic values of the vector y, $b_\mu \neq \breve{b}$. $\breve{b}$ is a zero of the function of b defined by the right-hand side of (14), and does not depend upon the values of y, while b_μ is defined by the condition

$$\sum_{k=1}^{K} (p_{1\mu}^k - p_{0\mu}^k) y^k = \varphi_1(b_\mu, \mu),$$

where $\varphi_b(b_\mu, \mu)$ denotes the partial derivative of the cost of effort function with respect to b, evaluated at the optimal effort, $\hat{b}(\mu)$.

Fix an information structure p: this determines $\breve{b}$. This determines also determines the minimum cost of inducing any effort level, $\varphi(b, \mu)$. Since the left-hand side of the equation defining b_μ is linear in $\mathbf{y}$, $b_\mu \neq \breve{b}$ for almost all values of $\mathbf{y}$. This completes the proof.

Proof of Lemma 5

Suppose that $\frac{p_{1G}^k}{p_{0G}^k} = \frac{p_{1B}^k}{p_{1B}^k}$ for all $k \in \{1, 2, ..K\}$. Let θ denote this common ratio. Fix an arbitrary signal, say K. Since $p_{\omega e}^K = 1 - \sum_{k=1}^{K-1} p_{e\omega}^k$,

$$\frac{1 - \sum_{k=1}^{K-1} p_{1G}^k}{1 - \sum_{k=1}^{K-1} p_{0G}^k} = \frac{1 - \sum_{k=1}^{K-1} p_{1B}^k}{1 - \sum_{k=1}^{K-1} p_{0B}^k}.$$

Multiplying by the two denominators and simplifying, we get

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k - \left(\sum_{k=1}^{K-1} p_{1G}^k\right)\left(\sum_{k=1}^{K-1} p_{0B}^k\right) = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k - \left(\sum_{k=1}^{K-1} p_{1B}^k\right)\left(\sum_{k=1}^{K-1} p_{0G}^k\right).$$

Using the fact that $p_{1G}^k p_{0B}^k = p_{1B}^k p_{0G}^k$ for $k \in \{1, 2.., K-1\}$, we get

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k - \left(\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1G}^k p_{0B}^j\right) = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k - \left(\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1B}^k p_{0G}^j\right).$$

By rewriting p_{1G}^k as θp_{1B}^k and p_{0G}^j as θp_{0B}^j, we get

$$\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1G}^k p_{0B}^j = \sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} \theta p_{1B}^k p_{0B}^j,$$

$$\sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} p_{1B}^k p_{0G}^j = \sum_{j\neq k, j=1}^{K-1} \sum_{k=1}^{K-1} \theta p_{1B}^k p_{0B}^j,$$

so that the above two expressions are equal. Hence

$$\sum_{k=1}^{K-1} p_{1G}^k + \sum_{k=1}^{K-1} p_{0B}^k = \sum_{k=1}^{K-1} p_{1B}^k + \sum_{k=1}^{K-1} p_{0G}^k,$$

which implies

$$p_{1G}^K + p_{0B}^K = p_{1B}^K + p_{0G}^K.$$

But since $p_{1G}^K p_{0B}^K = p_{1B}^K p_{0G}^K$, the two together imply either $p_{1G}^K = p_{1B}^K$ and $p_{0B}^K = p_{0G}^K$ or $p_{1G}^K = p_{0G}^K$ and $p_{0B}^K = p_{1B}^K$. Since the choice of signal K was arbitrary, this is true for every signal, contradicting Assumption 1*.

Proof of Theorem 2

The expected second period continuation value, $W(a, a^*)$, can be written as:

$$W(a,a^*) = \begin{cases} \sum_{y^k \in Y^D} p^k_{a\lambda} V(\pi^k_a, \mu^k_{a^*}) \text{ if } a < a^* \\ \sum_{y^k \in Y^U} p^k_{a\lambda} V(\pi^k_a, \mu^k_{a^*}) \text{ if } a > a^*. \end{cases}$$

The left-hand derivative of $W(a, a^*)$ at $a = a^* \in (0, 1)$, is given, after some simplification, by :

$$\left.\frac{\partial W^-(a,a^*)}{\partial a}\right|_{a=a^*} = \sum_{y^k \in Y^D} p^k_{a^*\lambda} V^+_\pi(\pi^k_{a^*}, \mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*}$$

which is strictly negative if Y^D is non-empty.

Similarly, the right-hand derivative is given by

$$\left.\frac{\partial W^+(a,a^*)}{\partial a}\right|_{a=a^*} = \sum_{y^k \in Y^U} p^k_{a^*\lambda} V^+_\pi(\pi^k_{a^*}, \mu^k_{a^*}) \left.\frac{\partial \pi^k_a}{\partial a}\right|_{a=a^*},$$

and this is strictly positive if Y^U is non-empty. Since either Y^U or Y^D is non-empty, we conclude that $\left.\frac{\partial W^+(a,a^*)}{\partial a}\right|_{a=a^*} > \left.\frac{\partial W^-(a,a^*)}{\partial a}\right|_{a=a^*}$.

The rest of the argument is identical to that in the proof of Theorem 1.

Appendix C: Proofs related to Theorem 3

Proof of Lemma 7

The partial derivatives for $\tilde{b}_y$ are both negative:

$$\frac{\partial \tilde{b}_y(a,h)}{\partial a} = \frac{[2y_h M_y - c'(\tilde{b}_y)]\rho\mu'_y}{M_y c''(\tilde{b}_y) - [c''(\tilde{b}_y)(1+\rho) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c'''(\tilde{b}_y)]\mu'_y h} \leq 0$$
$$\frac{\partial \tilde{b}_y(a,h)}{\partial h} = \frac{[c'(\tilde{b}_y) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c''(\tilde{b}_y)]\mu'_y}{M_y c''(\tilde{b}_y) - [c''(\tilde{b}_y)(1+\rho) + (\rho\tilde{b}_y + p_{0G} - p_{0B})c'''(\tilde{b}_y)]\mu'_y h} \leq 0.$$

Proof of Lemma 8

We begin with the observation that if $h(a)$ is increasing, then, by Lemma 7, $\tilde{b}_y(a, h(a))$ is decreasing in a (or, increasing in $\mu_y(a)$).

Note that

$$\Delta(\mu_y(a)) = \frac{y_h[M_y(a)]^2 + \mu'_y(a)h(a)c''(\tilde{b}_y(a,h(a)))(\rho\tilde{b}_y(a,h(a)) + p_{0G} - p_{0B})}{1 + \rho\frac{-\mu'_y(a)h(a)}{M_y(a)}}.$$

The denominator of this expression is decreasing and the numerator is increasing in $\mu_y(a)$.

Proof of Lemma 10

First, note that $\underline{a}_\Omega$ and $\overline{a}_\Omega$ are continuous in Ω and $\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega)$ depends on Ω only through $\underline{a}_\Omega$ (since $h_\Omega(\underline{a}_\Omega) = 0$), therefore is continuous in Ω as well.

Second, because $1/h_\Omega(a)$ is continuous in Ω for any a except, perhaps, $a = \underline{a}_\Omega$, $G_\Omega(a)$ is continuous in Ω for any a and $g_\Omega(a)$ is continuous in Ω everywhere except perhaps at $a = \underline{a}_\Omega$ (see Jeffery, 1925). Therefore, $G_\Omega(\underline{a}_\Omega)[\pi_1(\underline{a}_\Omega) + \pi_2(\underline{a}_\Omega)]$ is continuous in Ω.

Third, note that $\tilde{b}_y(a, h_\Omega(a))$ is continuous in Ω for any a except, perhaps, $a = \underline{a}_\Omega$, and therefore $[\pi_1(a) + \pi_2(a, \Omega)]g_\Omega(a)$ is continuous in Ω for every a except, perhaps $a = \underline{a}_\Omega$. Using Jeffery (1925), we can conclude that $\int_0^1 [\pi_1(a) + \pi_2(a, \Omega)]g_\Omega(a)da$ is continuous in Ω.