Online Appendix for
"Nonlinear Pricing of Storable Goods"

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Proof of Lemma II.1

Combining equations (2) and (3) we obtain:

$$P_1 + P_2 \leq 2\max_{s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \left(\underset{s \leq S}{Max}\{V(Q_1 - s) + V(s)\} + V(Q_2)\right).$$

Because V is concave, we have that

$$\max_{s \leq S}\{V(Q_1 - s) + V(s)\} + V(Q_2) \geq \underset{s \leq S}{Max}\{V(Q_1 - s) + V(Q_2 + s)\}.$$

Therefore,

$$P_1 + P_2 \leq \underset{s \leq S}{Max}\{V(Q_1 - s) + V(Q_2 + s)\}$$

which, is the same as inequality (1).

Proof of Theorem 2

Monopolist maximizes the sum of the tariffs, subject to all the constraints introduced in Section II.C. We start the proof by noticing, that some constraints can be omitted from the problem.

Lemma 0.1 *Constraints (3) and (5) are not binding:*

- *(4) and (9) imply (3);*

- *(2) and (7) imply (5).*

Proof. First we prove, that (4) and (9) imply (3). If we add up constraints (4) and (9), we obtain

$$P_1 \leq \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(Q_2 + s)\} + V(q_1),$$

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which is a tighter bound on P_1 than (3). Similarly, by adding up constraints (2) and (7) we obtain that

$$p_2 \leq V(q_2) + \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(s) + V(Q_2)\},$$

which implies (5). ■

Let us assume (and later prove) that inequalities (2), (4) and (7) are binding, i.e.

$$p_1 = V(q_1) \tag{1}$$

$$P_2 = \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(s)\} \tag{2}$$

$$p_2 = P_2 + V(q_2) - V(Q_2). \tag{3}$$

Given that, inequalities (9) and (11) provide the condition on P_1:

$$P_1 \leq p_1 + \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(Q_2 + s)\}$$

$$P_1 \leq p_1 + \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(s)\} - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(s)\}.$$

We also assume that one of these two inequalities is binding. By $Z(Q_1, q_1, x)$ we denote the following expression:

$$Z(Q_1, q_1, x) = \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(x + s)\} - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(x + s)\}$$

Using notation we obtain that

$$P_1 = p_1 + \min\{Z(Q_1, q_1, 0), Z(Q_1, q_1, Q_2)\}. \tag{4}$$

Equations (1),(2),(3) and (4) can be used to write down the profit of the monopolist as a function of q_1, q_2, Q_1 and Q_2.

$$\pi = V(q_1) + \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \leq s \leq S}\{V(Q_1 - s) + V(s)\}$$

$$+ \alpha(V(q_2) - V(Q_2)) + (1 - \alpha)\min\{Z(Q_1, q_1, 0), Z(Q_1, q_1, Q_2)\}$$

Observe, that $q_2^* = C^*$ and $Q_1^* = C^* + S$ maximize profit. To find the rest of the solution we need to maximize

$$V(q_1) + V(Q_2 + S) - \alpha V(Q_2) + (1 - \alpha)\min\{V(Q_2 + S)$$

$$- \max_{0 \leq s \leq S}\{V(q_1 - s) + V(Q_2 + s)\}, V(S) - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(s)\}\}$$

By proving the following lemma, we check, that the set of inequalities, that we assumed to be binding, are actually binding in the optimum.

Lemma 0.2 *Inequalities (2), (4), (7), (9) and (11) define the tariffs that maximize monopolist's profits.*

Proof. To prove this lemma we need to check if the rest of the inequalities are satisfied with the solution of the relaxed problem. In particular we need to check if inequalities (6), (8), (10) and (12). We start with (6)

$$
p_1 - P_1 - V(q_1) + V(Q_1) =
$$

$$
\max\left\{ \max_{0 \le s \le S}\left\{ \begin{matrix} V(q_1 - s) + V(Q_2 + s) \\ -V(Q_2 + S) - V(q_1) \end{matrix} \right\}, \max_{0 \le s \le S}\left\{ \begin{matrix} V(q_1 - s) + V(s) \\ -V(S) - V(q_1) \end{matrix} \right\} \right\} \le 0
$$

(8) is equivalent to

$$
p_1 + p_2 - P_1 - P_2 + \max_{0 \le s \le S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \le s \le S}\{V(q_1 - s) + V(q_2 + s)\} \ge
$$

$$
\max_{0 \le s \le S}\{V(q_1 - s) + V(Q_2 + s)\} - V(Q_2) - V(q_1) \ge 0
$$

(10) is equivalent to

$$
p_2 - P_2 + \max_{0 \le s \le S}\{V(Q_1 - s) + V(Q_2 + s)\} - \max_{0 \le s \le S}\{V(Q_1 - s) + V(q_2 + s)\} =
$$

$$
V(Q_2 + s) - V(Q_2) \ge 0
$$

and finally (12) is also satisfied:

$$
p_2 - P_1 - P_2 + \max_{0 \le s \le S}\{V(Q_1 - s) + V(Q_2 + s)\} - V(q_2) \ge
$$

$$
\max_{0 \le s \le S}\{V(q_1 - s) + V(Q_2 + s)\} - V(q_1) - V(Q_2) \ge 0.
$$

∎

Lemma 0.2 ensures, that the solution of a relaxed maximization problem coincides with the solution of the original profit maximization problem. Now that we have this result, we can get back to solving the relaxed problem. Let us consider the following expression:

$$
\min\{V(Q_2 + S) - \max_{0 \le s \le S}\{V(q_1 - s) + V(Q_2 + s)\}, V(S) - \max_{0 \le s \le S}\{V(q_1 - s) + V(s)\}\} \quad (5)
$$

If $Q_2 \geq q_1$ we have

$$\max_{0 \leq s \leq S}\{V(Q_2 + S) + V(q_1 - s) + V(s)\} - V(q_1) - V(Q_2) - V(S) \leq 0$$

and hence (5) becomes

$$\min\{V(Q_2 + S) - V(q_1) - V(Q_2), V(S) - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(s)\}\} =$$

$$V(Q_2 + S) - V(q_1) - V(Q_2).$$

If $Q_2 + 2S \geq q_1 \geq Q_2$ we observe, that

$$\max_{0 \leq s \leq S}\{V(Q_2 + S) + V(q_1 - s) + V(s)\} - 2V\left(\frac{q_1 + Q_2}{2}\right) - V(S) \leq 0$$

and (5) can be rewritten as

$$\min\{V(Q_2 + S) - 2V\left(\frac{q_1 + Q_2}{2}\right), V(S) - \max_{0 \leq s \leq S}\{V(q_1 - s) + V(s)\}\} =$$

$$V(Q_2 + S) - 2V\left(\frac{q_1 + Q_2}{2}\right).$$

Finally, if $Q_2 + 2S \leq q_1$ (5) becomes

$$\min\{-V(q_1 - S), -V(q_1 - S)\} = -V(q_1 - S).$$

We can rewrite part of maximization problem that solves for Q_2 and q_1 as

$$\max_{Q_2, q_1}\{f(q_1, Q_2)\}$$

where

$$f(q_1, Q_2) = \begin{cases} \alpha V(q_1) + (2 - \alpha)V(Q_2 + S) - V(Q_2), & \text{if } q_1 \leq Q_2 \\ V(q_1) + (2 - \alpha)V(Q_2 + S) - \alpha V(Q_2) \\ \quad - 2(1 - \alpha)V\left(\frac{q_1 + Q_2}{2}\right), & \text{if } Q_2 < q_1 \leq Q_2 + 2S \\ V(q_1) + V(Q_2 + S) - \alpha V(Q_2) - (1 - \alpha)V(q_1 - S), & \text{if } Q_2 + 2S < q_1 \end{cases}$$

First we observe that the solution for maximization problem can never satisfy $q_1 < Q_2$ because function $\alpha V(q_1) + (2 - \alpha)V(Q_2 + S) - V(Q_2)$ has unique maximum $q_1 = C^*$ and $Q_2 + S < C^*$. It means, that we should look for the solution in the set where $q_1 \geq Q_2$. There, however, observe that first order condition for q_1 and Q_2 suggest that $V'(q_1) > 0$ and

$V'(Q_2 + S) > 0$, hence $q_1 < C^*$ and $Q_2 < C^* - S$.

First order conditions that define Q_2^* and q_1^* are

$$V'(q_1^*) = \begin{cases} (1-\alpha)V'\left(\frac{q_1^* + Q_2^*}{2}\right) & \text{, if } Q_2^* \leq q_1^* < Q_2^* + 2S \\ (1-\alpha)V'(q_1^* - S) & \text{, otherwise} \end{cases}$$

and

$$V'(Q_2^* + S) = \begin{cases} \alpha V'(Q_2^*) + (1-\alpha)V'\left(\dfrac{q_1^* + Q_2^*}{2}\right) & \text{, if } Q_2^* \leq q_1^* < Q_2^* + 2S \\ \quad - (1-\alpha)V'(Q_2^* + S) & \\ \alpha V'(Q_2^*) & \text{, otherwise} \end{cases}$$

Proof of Theorem 3

Fix the time horizon to be T. Suppose the monopolist offers the flow q_t and charges p_t for it. Then consumer's problem is

$$\max_{c_t \geq 0, b_t \in \{0,1\}} \left\{ \int_0^T (V(y_t c_t) - x_t b_t p_t) \, dt \right\}$$

$$s_t = \int_0^t (x_\tau b_\tau q_\tau - y_\tau c_\tau) d\tau$$

$$x_t \in \begin{cases} 0, \text{ if } s_t = S \text{ and } b_t q_t - y_t c_t > 0 \\ \{0,1\}, \text{ otherwise} \end{cases}$$

$$y_t \in \begin{cases} 0, \text{ if } s_t = 0 \text{ and } b_t q_t - y_t c_t < 0 \\ \{0,1\}, \text{ otherwise} \end{cases}$$

Where vector (x_t, y_t) denotes so-called regime. The meaning of this regime variables in our problem is the following. By setting variable $x_t = 0$ we make sure, that when the agent's storage is filled up to maximum capacity, the agent does not purchase the flow of good that is larger than his consumption. Similarly, by setting $y_t = 0$, we guarantee, that when the agent's storage is empty, the agent can not consume more than what he purchases. Note, that correspondence that defines the domain of x_t and y_t is right continuous.

The control variables in this problem are c_t and b_t. By b_t we denote a binary decision whether consumer buys a flow at time t or not. Naturally, by c_t we denote consumption at time t.

The state variable for this problem is the amount of good, that is stored in the consumer's inventories at time t, i.e. s_t.

By H we denote Hamiltonian for this problem:

$$H = \Psi_t(x_t b_t q_t - y_t c_t) + V(y_t c_t) - x_t b_t p_t.$$

Following Panteleev et al. (2011)[1], we obtain necessary conditions for this problem:

$$c_t^* = \begin{cases} 0, \text{ if } y_t^* = 0 \\ V'^{-1}(\Psi_t^*), \text{ otherwise} \end{cases}$$

$$b_t^* \in \begin{cases} 1, \text{ if } x_t^* = 1 \text{ and } \Psi_t^* > \frac{p_t}{q_t} \\ \{0,1\}, \text{ if } x_t^* = 1 \text{ and } \Psi_t^* = \frac{p_t}{q_t} \\ 0, \text{ otherwise} \end{cases}$$

Note, that Ψ_t^* is piecewise constant with jumps at discontinuity points of x_t^* and y_t^*.

These necessary conditions state, that agents smooths his consumption whenever he has some amount of good in the storage. Also, the agent purchases the good only when per unit price is lower than the marginal utility of his current consumption.

Lemma 0.3 *If $c_\tau^* = 0$, it must be that $c_t^* = 0$ and $b_t^* = 0$ for all $t \in [0, \tau]$*

Proof. By contradiction let us assume that $c_t^* > 0$ for all $t \in [t_1, t_2] \subset [0, \tau]$ (since Ψ_t^* is piecewise constant there must exist non-degenerate interval). Then it must be that $\int_0^{t_2} b_t^* dt > 0$. We can always find $\epsilon > 0$ small enough, such that consumer stores ϵ more by the time t_2 and consumes it around time τ. Since $c_\tau^* = 0$ and $V(\cdot)$ is concave, it is an improvement, hence the contradiction. ■

By this Lemma, we can restrict our attention on policies that induce strictly positive consumption everywhere.

Let us partition our time interval $[0, T]$ into intervals $\{[t_{i-1}, t_i]\}_{i=1}^I$ such that $t_0 = 0$, $t_I = T$, and for all $1 \le i < I$: $t_i = t \iff s_t^* = S$ and for any $\epsilon > 0$ s_τ^* is not constant on $\tau \in (t - \epsilon, t + \epsilon)$. By construction, consumption inside interval i is constant if $\forall t \in [t_{i-1}, t_i] : s_t > 0$ (we denote the consumption in the interval i by c_i in this case). Aggregate amount of good purchased within the interval i is

$$\int_{t \in [t_{i-1}, t_i]} x_t^* b_t^* q_t dt = \begin{cases} c_i(t_i - t_{i-1}), \text{ if } 1 < i < I \\ c_1 t_1 + S, \text{ if } i = 1 \\ c_I(T - t_{I-1}) - S, \text{ if } i = I \end{cases}$$

[1]See Panteleev, A.V., Bortakovskiy, A.S., Letova, T.A., "Optimalnoe Upravlenie V Primerah I Zadachah", Izdatelstvo MAI, 1996.

Now let us look at the intervals that have the property that $\exists t \in (t_{i-1}, t_i) : s_t^* = 0$. Take such interval $[t_{i-1}, t_i]$ if it exists and partition it further into subintervals, such that consumption is constant within each subinterval (we can do that because Ψ_t^* is piecewise constant). Lets index those subintervals by $j = 1, ..., m_i$. The endpoints of those intervals are $t_i^0 = t_{i-1}$, and $\{t_i^j\}_{j=1}^{m_i}$, where t_i^j is the right endpoint of jth interval. Observe, that consumption is increasing in j i.e. $c_i^j < c_i^{j+1}$ for all $j = 1, ..., m_i - 1$. Also the aggregate amount of good purchased within each subinterval is

$$\int_{t \in [t_i^{j-1}, t_i^j]} x_t^* b_t^* q_t dt = \begin{cases} c_i^j(t_i^j - t_i^{j-1}), \text{ if } 1 < j < m_i \\ c_i^1(t_i^0 - t_i^1) - S, \text{ if } j = 1 \\ c_i^{m_i}(t_i - t_i^{m_i-1}) + S, \text{ if } j = m_i \end{cases}$$

From necessary conditions we know that per unit price of a good is bounded from above by $V'(c)$. Also, we know that $V'(c_i^{m_i}) \leq V'(c_i^1)$, hence the profits of the monopolist that are collected from sales in interval i are bounded from above by

$$\sum_{j=1}^{m_i}(t_i^{j-1} - t_i^j)c_i^j V'(c_i^j) + S(V'(c_i^{m_i}) - V'(c_i^1)) \leq \sum_{j=1}^{m_i}(t_i^{j-1} - t_i^j)c_i^j V'(c_i^j)$$

Now let us index our partitions by $k \in K$ such that the new partition is the coarsest refinement of the partitions above. Again price of a good is bounded from above by $V'(c)$ so total profits that are bounded from above by

$$\sum_{k \in K}\frac{(t_k - t_{k-1})}{T}c_k V'(c_k) + \frac{S}{T}(V'(c_1) - V'(c_I))$$

In the limit this bound becomes

$$\limsup_{T \to \infty}\left(\sum_{k \in K}\frac{(t_k - t_{k-1})}{T}c_k V'(c_k) + \frac{S}{T}(V'(c_1) - V'(c_I))\right) \leq \max_{c \geq 0}\{cV'(c)\}$$

The expression on the right hand side is the profit from pricing the good linearly.

Proof of Lemma A.1

We now need to prove that it is indeed optimal to induce binding storage constraints. There are several cases to be considered.

Consider first, the case in which the monopolist sets a policy in which storage is interior: $0 < s < S$. We first discuss the case in which storage does not bind even if the consumer chooses to skip the second period purchase, i.e., $\frac{Q_1}{2} \leq S$. In this case, consumer optimal

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smoothing behavior implies that profits are:

$$\Pi^I = 4V\left(\frac{Q_1 + Q_2}{2}\right) - 2V\left(\frac{Q_1}{2}\right) - V(Q_2).$$

The first order conditions can be combined to yield

$$2V'\left(\frac{Q_1^I + Q_2^I}{2}\right) = V'\left(\frac{Q_1^I}{2}\right) = V'(Q_2^I)$$

so that

$$\frac{Q_1^I}{2} = Q_2^I \text{ and } 2V'\left(\frac{3Q_2^I}{2}\right) = V'(Q_2^I).$$

For storage not to be binding even when the consumer chooses to skip the second period purchase, it must be the case that $S \geq Q_2^I$.

We now show that all policies in the interior of this class violate the second order conditions for the monopolist.

Assume by way of contradiction that Q_1^I and Q_2^I satisfy the first and second order conditions, namely

$$V''\left(\frac{Q_1^I + Q_2^I}{2}\right) - V''(Q_2^I) \leq 0$$

$$\frac{1}{2}V''(Q_2^I)\left(V''(Q_2^I) - 3V''\left(\frac{Q_1^I + Q_2^I}{2}\right)\right) \geq 0$$

These two inequalities imply that

$$2V''\left(\frac{Q_1^I + Q_2^I}{2}\right) \leq V''(Q_2^I) \leq 3V''\left(\frac{Q_1^I + Q_2^I}{2}\right)$$

which can only be true if $V''\left(\frac{Q_1^I + Q_2^I}{2}\right) \geq 0$. This contradicts the assumption that V is strictly concave showing the desired contradiction.

This means that the solution to the maximization problem must be on the boundary of this set: one of the constraints on storage is binding.

We now need to consider the case where capacity binds for skipping the second period purchase but not for smoothing consumption. The reasoning is very similar. In this case, profits are

$$\Pi^{II} = 4V\left(\frac{Q_1 + Q_2}{2}\right) - V(Q_1 - S) - V(S) - V(Q_2).$$

First order conditions for this problem imply that $Q_1^{II} - S = Q_2^{II}$. Second order condition

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then is

$$V''\left(\frac{Q_1^{II} + Q_2^{II}}{2}\right) - V''(Q_2^{II}) \leq 0$$

$$V''(Q_2^{II})\left(V''(Q_2^{II}) - 2V''\left(\frac{Q_1^{II} + Q_2^{II}}{2}\right)\right) \geq 0$$

By combining two inequalities together we get that

$$V''\left(\frac{Q_1^{I} + Q_2^{I}}{2}\right) \leq V''(Q_2^{I}) \leq 2V''\left(\frac{Q_1^{I} + Q_2^{I}}{2}\right)$$

but this contradicts the concavity of V.

Finally, we need to show that the seller does not prefer to sell $Q_1 < Q_2$ in which case optimal storage would be zero. When the seller sets $Q_1 < Q_2$ (and as long as capacity binds in the event that the consumer skips second period purchases), there are two possibilities: in the first case, when $Q_1 \geq 2S$, if the consumer skips the second period purchase, then storage capacity binds. In this case, profits are

$$\Pi^0 = 2(V(Q_1) + V(Q_2)) - V(Q_1 - S) - V(S) - V(Q_2)$$

which are maximized when

$$2V'(Q_1^0) = V'(Q_1^0 - S)$$
$$V'(Q_2^0) = 0$$

thus we can rewrite profits as:

$$2V(Q_1^0) + V^* - V(Q_1^0 - S) - V(S).$$

It is easy to see that these are the same profits as in our candidate optimal policy. The role of Q_1 and Q_2 are now reversed: second period consumption is efficient while first period consumption is inefficiently low.[2] This only happens when capacity is small enough, i.e. $S \leq \tilde{S}$ where $\tilde{S}$ solves

$$2V'(2\tilde{S}) = V'(\tilde{S}).$$

When $S \geq \tilde{S}$, capacity does not bind when the consumer skips second period purchases.

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[2]Thus, when storage capacity is low there is another solution. We do not highlight this solution because it is no longer optimal in the cases considered later.

Observe that $Q_1 < 2S$ in this case. Monopolist profits are then

$$\max_{Q_1}\{V^* + 2V(Q_1) - 2V\left(\frac{Q_1}{2}\right)\}.$$

We need to show, that

$$\max_{Q_1}\{V^* + 2V(Q_1) - 2V\left(\frac{Q_1}{2}\right)\} < \Pi^S$$

We notice that if $Q_1 \geq S$ we can set $Q_2 = Q_1 - S$ and obtain

$$2V(Q_1) - 2V\left(\frac{Q_1}{2}\right) < 2V(Q_2 + S) - V(Q_2) - V(S) \leq \Pi^S$$

and, if $Q_1 < S$, we obtain that

$$2V(Q_1) - 2V\left(\frac{Q_1}{2}\right) < V(Q_1) < V(S) \leq \Pi^S$$

Proof of Lemma A.3

This lemma is almost identical to Theorem 3, so we only sketch the proof here.

Suppose by contradiction, that $V'(x_t) > \frac{V(q_t)}{q_t}$, and agent's storage is not full, i.e. $s_t < S$. By purchasing multiple flow bundles, the agent can buys a measure $\epsilon > 0$ of good (where ϵ is small enough)[3], he is going to pay $\frac{\epsilon V(q_t)}{q_t}$. After the purchase agent can spread this small portion of the good across $\sqrt{\epsilon}$ of time. His consumption is going to go up by $\frac{\epsilon}{\sqrt{\epsilon}} = \sqrt{\epsilon}$. The net gain in utility in this case is

$$\sqrt{\epsilon}\left(\sqrt{\epsilon}V'(x_t)\right) - \frac{\epsilon V(q_t)}{q_t} > 0$$

which is a desired contradiction. The same logic works for the case, when $V'(x_t) < \frac{V(q_t)}{q_t}$.

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[3]This can be achieved by setting $B_t - \lim\limits_{\tau \to t-0} B_\tau = \frac{\epsilon}{q_t}$.