Online Appendix for "Competition and Networks of Collaboration"

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A1 Alternative model

This section contains an extension to the main model that considers the case of costless links and benefits from indirect collaboration. The setup shares many common features with the main model. The following is an incomplete description of the model; please see the paper for further details.

I assume that bilateral collaboration requires the consent of both partners. For instance, if collaboration between two agents exists and one decides to quit, the collaboration is terminated immediately. For simplicity, the collaboration is assumed to have a unit intensity.

An outcome in this model is a pair (F, T), where $F \in \{0,1\}^{N\times N}$ is an adjacency matrix that describes a network of collaboration and $T \in \mathbb{R}_+^{N\times N}$ is a matrix that describes a system of transfers between the agents.

For any outcome (F, T), matrix F is assumed to be symmetric—i.e., it is assumed that collaboration is mutual. $F_{i,j} = 1$ means that agents i and j are collaborating with each other. The following notation will be useful: For $M \subset N, I(M) \in \{0,1\}^{N\times N}$ is an adjacency matrix, such that for all $i \neq j : [I(M)]_{i,j} = 1$ if $\{i,j\} \subset M$ and $[I(M)]_{i,j} = 0$ otherwise. In particular,

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matrix $I(\emptyset)$ describes the empty network and $I(N)$ describes the complete one. For two matrices X and Y, I denote their Hadamard product by $X \circ Y$: $\forall i, j : [X \circ Y]_{i,j} = X_{i,j} Y_{i,j}$.

Matrix T describes the transfers between the agents. I assume that $T_{i,j} \geq 0$ is the amount agent i pays to agent j in the outcome (F, T). I denote an outcome with zero transfers by $(F, 0)$. Finally, I denote a set of all feasible outcomes by $\mathcal{U}$.

Given a network of collaboration F, each agent i produces the output $y(F, i)$ that depends on the amount of collaboration in which this agent is involved:

$$y(F, i) = \sum_{k=0}^{\infty} \sum_{j=1}^{n} \alpha_k (F^k)_{ji},$$

where α_k represents the contribution of the indirect collaboration with the agents that are k connections away from agent i. I normalize $\alpha_0 = \alpha_1 = 1$ and assume that α_k is decreasing in k. There are two special cases: (i) when $\alpha_k = 0$ for all $k > 1$, the output is equal to the degree of the agent in F; and (ii) when $\alpha_k = \alpha^k$, the output is equal to the Katz centrality measure of node i in network F.

Similar to the main model, the agents' payoffs are additive in a tournament prize, a value of collaboration and transfers:

$$U_i(F, T) = r\left(p_i(F), q_i(F)\right) + f\left(y(F, i)\right) + \sum_{j=1}^{n} T_{j,i} - \sum_{j=1}^{n} T_{i,j},$$

where

$$r(i, j) = \frac{1}{j - i + 1} \sum_{k=i}^{j} R(k).$$

p_i and q_i denote the lower and the upper bounds on possible rankings for agent i in the tournament. These bounds are defined as follows:

$$p_i(F) = |\{k \in N : y(F, i) < y(F, k)\}| + 1$$

and

$$q_i(F) = n - \left|\{k \in N : y(F,i) > y(F,k)\}\right|.$$

By $\mathbf{U}_M(F,T)$, I denote a vector of utilities for the set of agents M in outcome (F,T). Also, for two vectors $\mathbf{U}_M, \mathbf{V}_M$ I say that $\mathbf{U}_M \gg \mathbf{V}_M$ if $\forall i \in M : U_i > V_i$.

Since the agents' utilities are linear in the transfers and f is strictly increasing, the set of efficient outcomes consists of all outcomes in which all agents collaborate at the maximum level.

Remark A1.1. *An outcome (F,T) is efficient if and only if $F = I(N)$—i.e., if a corresponding network of collaboration is complete.*

Proof. Start with the observation that the Pareto frontier is a straight line with a slope of 45 degrees. Therefore, one can use the utilitarian welfare criterion. Consider an outcome (F,T). Observe that $\sum_{i=1}^{n}\sum_{j=1}^{n} T_{i,j} = 0$. The social welfare in this outcome is

$$W = \sum_{i=1}^{n} U_i(F,T) = \sum_{i=1}^{n} f\left(y(F,i)\right) + \sum_{i=1}^{n} R(i).$$

The social welfare is strictly increasing in F and does not depend on T. □

The stability notion used for this extension of the model is the same as in the main model with some minor changes in the following definition.

Definition A1.2. *A coalition M can enforce a transition from outcome (F,T) to outcome (F',T')—i.e., $(F,T) \overset{M}{\to} (F',T')$ if for all $i,j \in N$:*

(i) $F'_{i,j} > F_{i,j}$ *or* $T'_{i,j} > T_{i,j}$ *implies* $i,j \in M$*; and*

(ii) $F'_{i,j} < F_{i,j}$ *or* $T'_{i,j} < T_{i,j}$ *implies* $i \in M$ *or* $j \in M$*.*

A1.1 Analogs of the main results

As in the main model, in this extension I also define the group optimal sequences—i.e., the outcomes that have group structure and replicate the main results of the paper.

Definition A1.3. *Consider a sequence* $\{m_k\}_{k=1}^K$. *Let* $M_0 = 0$, *and for* $k \geq 1$, *let* $M_k = \sum_{i=1}^{k} m_i$. *The sequence* $\{m_k\}_{k=1}^K$ *is group-optimal if* $M_K = n$, *and for all* $k \geq 1$

$$m_k \in \underset{\frac{n-M_{k-1}}{2} < m \leq n - M_{k-1}}{\arg\max} \left\{ r(1 + M_{k-1}, m + M_{k-1}) + f(m) \right\}.$$

Given a group-optimal sequence $\{m_k\}_{k=1}^K$, let

$$V_k = r\left(1 + M_{k-1}, M_k\right) + f\left(m_k - 1\right).$$

Definition A1.4. *A collaborative network* F *has a group structure induced by the sequence* $\{m_k\}_{k=1}^K$ *if there exists a partition* $\mathcal{N} = \{N_1, ..., N_K\}$ *of the set* N *such that*

(i) $\forall k : |N_k| = m_k$; *and*

(ii) $F = \sum_{i=1}^{K} I(N_i)$.

Theorem A1.5. *Suppose that agents cannot use transfers—i.e., the set of feasible outcomes is* $\mathcal{U}_0 = \{(F, 0_{n,n}) \mid F \in \{0,1\}^{N \times N}\}$. *A set of all outcomes* $(F, 0) \in \mathcal{U}_0$ *in which* F *has the group structure induced by the group optimal sequence is farsighted stable.*

Proof.

Definition A1.6. *An outcome* $\gamma = (F, T)$ *contains a semi-component formed by a set of agents* M *if for all* $i \in M : F_{i,j} = \mathbb{I}\{j \in M\}$, $\sum_{j \in M} T_{i,j} = \sum_{j \in M} T_{j,i}$, *and* $\sum_{j \notin M} T_{i,j} = 0$.

Lemma A1.7. *Let* $M \subset N : |M| \geq N/2$. *For any outcome* (F, T) *such that* $F_{i,j} = 0$ *for all* $i, j : |\{i, j\} \cap M| = 1$ *and* $T_{j,i} = 0$ *for all* $i \in M, j \in N \setminus M$ *the following holds:*

$$\min_{i \in M} U_i(F, T) \leq r(1, |M|) + f(|M|).$$

Proof. Let $\mathcal{G}$ be a set of all outcomes that satisfy the conditions of the lemma: $\mathcal{G} = \{(F,T) \in \mathcal{U} \mid F_{i,j} = 0 \text{ and } T_{i,j} = 0 \text{ for all } i,j : |\{i,j\} \cap M| = 1\}$. The maximum sum of the payoffs of all agents in M across outcomes in $\mathcal{G}$ is

$$\max_{(F,T)\in\mathcal{G}} \sum_{i\in M} U_i(F,T) = |M|f(|M|) + |M|r(1,|M|),$$

hence

$$\min_{i\in M} U_i(F,T) \leq \frac{1}{|M|}\sum_{i\in M} U_i(F,T) \leq r(1,|M|) + f(|M|)$$

□

Lemma A1.8. *Denote a set of agents whose payoff is below* V_1 *by* $A(F,0) = \{i : U_i(F,0) < V_1\}$. *For any outcome* $(F,0)$ *such that* $|A(F,0)| < n/2$, *either* F *contains a semi-component of size* m_1 *or one can always find* $(F',0)$ *such that*

(i) $(F,0) \overset{A(F,0)}{\to} (F',0)$;

(ii) $A(F,0) \subset A(F',0)$;

(iii) $\sum_{j\in N} \sum_{i\in A(F,0)} F_{i,j} > \sum_{j\in N} \sum_{i\in A(F,0)} F'_{i,j}$; *and*

(iv) F' *does not contain a semi-component of size* m_1.

Proof. Suppose that F does not contain a semi-component of size m_1. By Lemma A1.7, there exists $j \notin A(F,0)$ such that

$$\sum_{i\in A(F,0)} F_{i,j} > 0.$$

Let $\tilde{F}$ be such that $\tilde{F}_{i,j} = F_{i,j}\mathbb{I}\{i,j \notin A(F,0)\}$. There are two cases to consider: (i) $\tilde{F}$ contains a complete component of size m_1 and (ii) the opposite.

In case (ii), $F' = \tilde{F}$ satisfies the conditions of the lemma.

Consider case (i), in which $\tilde{F}$ contains a complete component of size m_1. It must be that there are at least two links that are in F and not in $\tilde{F}$. Indeed, if there is only one such link, by convexity of R, the payoff of everyone else in the set $N \setminus A(F,0)$ in the outcome F is

$$r(2,m_1)+f(m_1+\alpha_1)<r(1,m_1+1)+f(m_1+1)\leq V_1$$

which is a contradiction. Let one of these links be between agents k and l. Consider $\hat{F}$, such that

$$\hat{F}_{i,j}=F_{i,j}\mathbb{I}\{i,j\notin A(F,0)\}+F_{i,j}\mathbb{I}\{\{i,j\}=\{k,l\}\}.$$

The outcome $(\hat{F},0)$ satisfies the conditions of the lemma. $\square$

Let $\mathcal{R}$ be a set of all outcomes $(F,0)\in\mathcal{U}$ in which F has a group structure induced by a group-optimal sequence.

I show that the set $\mathcal{R}$ satisfies internal and external stability. I show that for any $(F,0),(G,0)\in\mathcal{R}$, $(F,0)\not\triangleright(G,0)$. Let $\mathcal{H}=\{H_1,...\}$ be a partition that induces a network of collaboration F and $\mathcal{G}=\{G_1,...\}$ be a partition that induces G. Also, let $K=\{i\in N: U_i(F,0)>U_i(G,0)\}$. Denote an index of the largest set infiltrated by agents from K in F by k—i.e., for all $j<k: K\cap H_j=\emptyset$ and $K\cap H_k\neq\emptyset$. Let $M=\bigcup_{j\leq k}G_j$, and note that $|M|>n/2$. For any $S\subset N\setminus M$ and for any $(F',0):(G,0)\overset{S}{\to}(F',0)$, we have $U_M(G,0)=U_M(F',0)$. Hence, if $(F,0)\triangleright(G,0)$, it must be that $U_M(G,0)=U_M(F,0)$, which contradicts $K\cap M\neq\emptyset$.

I show external stability by construction. For every state $(F,0)\notin\mathcal{R}$ I find $(G,0)\in\mathcal{R}:(G,0)\triangleright(F,0)$.

Consider $(F,0)\notin\mathcal{R}$, such that F does not contain a complete component of size m_1. By Lemma A1.7, a set $A(F,0)=\{i: U_i(F,0)<V_1\}$ is nonempty. Moreover, if $|A(F,0)|<n/2$, one can apply Lemma A1.8 repeatedly to obtain a sequence of outcomes $\{(F^i,0)\}$ such that the last element of the sequence, $(F^L,0)$, satisfies $m_1\geq|A(F^L,0)|\geq n/2$.

Consider an outcome $(F^{L+1},0)$ such that

$$F^{L+1}=I(A(F^L,0))+F^L\circ I(N\setminus A(F^L,0)).$$

Clearly, $N = A(F^{L+1}, 0)$. Take a set $N_1 : A(F^L, 0) \subset N_1$ and $|N_1| = m_1$ and consider an outcome $(F^{L+2}, 0)$ such that $F^{L+2} = I(N_1) + F^{L+1} \circ I(N \setminus N_1)$. This procedure obtains a sequence of outcomes $(F^i, 0)$ and a sequence of coalitions S_i that satisfy the following properties for any $i \in \{1, ..., L+2\}$:

1. $S_i \subset N_1$;
2. $(F^i, 0) \xrightarrow{S_i} (F^{i+1}, 0)$; and
3. $\forall j \in S_i : U_j(F^i, 0) < U_j(F^{L+2}, 0) = V_1$.

In this part of the sequence, the largest group—i.e., the group of size m_1—forms a complete component. The rest of the sequence is constructed by induction: Suppose there exists a sequence along which the largest k groups form complete components. I use the argument above to construct part of the sequence, along which $k+1$th largest group forms a component. There exists $N_{k+1} \subset N \setminus \bigcup_{j \leq k} N_j : |N_{k+1}| = m_{k+1}$ such that this part of the sequence, enumerated by $i \in \{I_k + 1, ..., I_{k+1}\}$, satisfies the following three conditions for any $i \in \{I_k + 1, ..., I_{k+1}\}$:

1. $S_i \subset N_{k+1}$;
2. $(F^i, 0) \xrightarrow{S_i} (F^{i+1}, 0)$; and
3. $\forall j \in S_i : U_j(F^i, 0) < U_j\left(F^{I_{k+1}}, 0\right) = V_{k+1}$.

□

Theorem A1.9. *Suppose that agents cannot use transfers—i.e., the set of feasible outcomes is* $\mathcal{U}_0 = \{(F, 0_{n,n}) \mid F \in \{0,1\}^{N \times N}\}$. *Also, suppose that* $R(k) = R(n)$ *for all* $k > 1$.[1] *Then, there exists a stable set* $\mathcal{R}$ *that contains an efficient outcome if and only if*

$$n \in \underset{\frac{n}{2} < m \leq n}{\arg\max} \left\{ f(m) + r(1, m) \right\}. \tag{A1.1}$$

[1]This condition restricts the set of tournaments to winner-takes-all. It can be relaxed to $R(2) < r(1, n)$, which, roughly speaking, requires that R be very convex.

Proof. If (A1.1) holds, Theorem A1.5 implies that there exists a singleton stable set that contains the efficient outcome with the complete network of collaboration.

Suppose (A1.1) does not hold and let $(F, 0)$ be an outcome in which F has a group structure induced by some group-optimal sequence $\{m_k\}_{k=1}^{K}$. Note that $m_1 \neq n$. Recall that $I(N)$ is a complete network and let $\mathcal{R}$ be a farsighted stable set.

Assume by contradiction that $(I(N), 0) \in \mathcal{R}$. Since $(I(N), 0) \not\triangleright (F, 0)$ and $(F, 0) \triangleright (I(N), 0)$, it must be that $(F, 0) \notin \mathcal{R}$, and there must exist $(F', 0) \in \mathcal{R}$ such that $(F', 0) \triangleright (F, 0)$. Then, $(I(N), 0) \triangleright (F', 0)$, which is a contradiction.

To show this, define

$$H(G) = \left\{ i \in N \middle| \forall k \in N : y(G, i) \geq y(G, k) \right\}$$

There are two cases to consider: Either (i) $|H(F')| < n/2$ or (ii) $n > |H(F')| \geq n/2$.

Since $R(2) < r(1, n)$, for all $i \notin H(F')$

$$U_i(F', 0) \leq f\left(\sum_{j \in N} F'_{i,j}\right) + R(2) < f(n) + r(1, n) = U_i(I(N), 0). \qquad \text{(A1.2)}$$

In case (i), pick an arbitrary $k \in N \setminus H(F')$ and let F^1 be a network such that $F^1_{i,j} = F'_{i,j}$ for all $i, j \in H(F')$ and for all $i \in N \setminus (H(F') \cup \{k\}) : F^1_{i,j} = \mathbb{I}\{j = k\}$. Note that $F' \overset{N \setminus H(F')}{\rightarrow} F^1$ and $H(F^1) = \{k\}$. Let F^2 be a 2-regular network such that for all $i : F^1_{i,k} = 0 \implies F^2_{i,k}$—i.e., $F^1 \overset{N \setminus \{k\}}{\rightarrow} F^2$. Finally, $F^2 \overset{N}{\rightarrow} I(N)$. Observe that for all three transitions, by (A1.2), the payoffs of acting agents are strictly below $f(n) + r(1, n)$.

In case (ii), to construct a sequence of outcomes, I set up the induction. Let F^k be a network such that $H(F^k) > n/2$. I use Lemma A1.7 to establish that there exists an agent $i \notin H(F^k)$ such that

$$\sum_{j \in H(F^k)} F_{i,j} > 0.$$

Pick such an agent i and an agent $j : F^k_{i,j} = 1$. Construct $F^{k+1} = F^k - I(i,j)$. Note that $H(F^{k+1}) = H(F^k) \setminus \{j\}$.

Start with $F^0 = F'$ and construct the sequence using this recursion. Let F^K be the last well-defined element of the sequence (note that K is finite, because at some point in the sequence Lemma 1 is no longer applicable). There are two subcases to consider: $H(F^K) = n/2$ and agents in $H(F^K)$ form a component, or $H(F^K) < n/2$. In the latter case, we continue constructing the sequence using case (i) of this proof. In the former case, without loss of generality, let agents in $H(F^K)$ be indexed by odd numbers in N. Construct a network F^{K+1} that inherits the component formed by $H(F^K)$ in F^K in which all other agents mimic the same component—i.e., such that for all $i, j \in N$:

$$F^{K+1}_{i,j} = \begin{cases} F^K_{i,j}, \text{ if } i,j \in H(F^K), \\ F^K_{i+1,j+1}, \text{ if } i,j \notin H(F^K), \\ 0, \text{ otherwise.} \end{cases}$$

For any $k \in \{0, ..., K\} : F^k \overset{N \setminus H(F^k)}{\to} F^{k+1}$. Finally, $F^{K+1} \overset{N}{\to} I(N)$. At all of these transitions, the payoffs of acting agents are strictly below $f(n) + r(1,n)$. □

The condition (A1.1) is equivalent to

$$R(1) \leq \min_{m > \frac{n}{2}} \left\{ \frac{mn}{n-m}(f(n) - f(m)) \right\},$$

for winner-takes-all tournaments.

There are other farsighted stable sets in this model beyond the ones characterized in Theorem A1.5.

Theorem A1.10. *Suppose that agents cannot use transfers—i.e., the set of feasible outcomes is* $\mathcal{U}_0 = \{(F, 0_{n,n}) \mid F \in \{0,1\}^{N \times N}\}$. *Also, suppose that* $n \geq 5$, $\alpha_k = 0$ *for all* $k > 1$, *and let* $m^* \in \mathbb{N} : n/2 < m^* \leq n/2 + 1$ *be the size of the smallest majority. If* $R(1) > R(n)$ *and* $R(k) = R(n)$ *for all* $k > 1$, *an outcome* $(F, 0)$ *such that* *(i)* $\forall i : i \leq m^* : \sum_{j=1}^{n} F_{i,j} = n-1,$*; and*

(ii) $\forall i : m^* < i < n : \sum_{j=1}^{n} F_{i,j} = n-2$

is a singleton farsighted stable set.

Proof. The internal stability of $\{(F, 0)\}$ is trivially satisfied. To show that this singleton set is externally stable, consider an arbitrary outcome $(F', 0)$. Let

$$L(F') = \{i \in N \mid U_i(F', 0) < U_i(F, 0)\}\}$$

and

$$H(F) = \left\{ i \in N \middle| \forall k \in N : \sum_{j \in N} F_{i,j} \geq \sum_{j \in N} F_{k,j} \right\}$$

For any $i : U_i(F) > f(n-2)$. Note that $L(F')$ is not empty whenever $F' \neq F$. Also, if $\{i\} = L(F')$, then there exists $F'' : F \overset{\{i\}}{\to} F'', L(F') \subset L(F'')$ and $|L(F'')| \geq 2$.

Construct a sequence using the following induction: Start with $F^0 = F''$. If $L(F^k) < n/2$, there exists at least one agent $i \in L(F^k)$ and another agent $j \notin L(F^k)$ such that $F^k_{i,j} = 1$ (otherwise, recall that $|L(F^k)| \geq 2$; hence, for all agents $j \notin L(F^k) : \sum_{i \in N} F^k_{j,i} \leq n-3$, which implies that at least one of these agents should be in $L(F^k)$). In this case, construct $F^{k+1} = I(N \setminus L(F^k)) \circ F^k$. Clearly, $F^k \overset{L(F^k)}{\to} F^{k+1}$ and $L(F^k) \subsetneq L(F^{k+1})$. Let F^K be the last well-defined element of this sequence: $L(F^K) \geq n/2$.

Let $M \subset L(F^K) : n/2 \leq |M| < n/2 + 1$. Consider $F^{K+1} = I(M) + I(N \setminus M) \circ F^K$, $F^{K+2} = I(H(F)) + I(N \setminus H(F)) \circ F^{K+1}$ and $F^{K+3} = I(H(F)) + I(N \setminus H(F)) \circ F$. Note that $F^K \overset{M}{\to} F^{K+1} \overset{H(F)}{\to} F^{K+2} \overset{N \setminus H(F)}{\to} F^{K+3} \overset{N}{\to} F$. Also, $\mathbf{U}_{H(F)}(F^{K+1}) \ll \mathbf{U}_{H(F)}(F)$, $\mathbf{U}_{N \setminus H(F)}(F^{K+2}) \ll \mathbf{U}_{N \setminus H(F)}(F)$ and $\mathbf{U}_N(F^{K+3}) \ll \mathbf{U}_N(F)$.

□

In the stable set described in Theorem A1.10, the smallest majority is fully connected and the largest minority minimally handicaps itself to sustain top

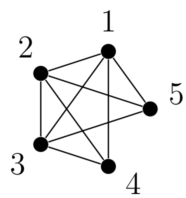

Figure 1: An example of a singleton farsighted stable set

rankings for the majority. Figure 1 depicts an example in which agents 1, 2, and 3 form the fully connected smallest majority and agents 4 and 5 handicap themselves by not collaborating with each other.

Theorem A1.11. *A set of outcomes $\mathcal{R}$ that consists of all $(F,T) \in \mathcal{R}$, such that*

(i) *F has a group structure induced by a group-optimal sequence $\{m_k\}_{k=1}^{K}$;*

(ii) *$F_{i,j} = 0$ implies $T_{i,j} = 0$; and*

(iii) *for any $i \in N$, $\sum_{j \in N} T_{i,j} = \sum_{j \in N} T_{j,i}$*

is farsighted stable.

Proof.

Lemma A1.12. *Denote a set of agents whose payoff is below V_1, by $A(F,T) = \{i : U_i(F,T) < V_1\}$. For any outcome $(F,T) : |A(F,T)| < n/2$, either F contains a semi-component of size m_1 or one can always find (F',T'), such that*

(i) *$(F,T) \overset{A(F,T)}{\rightarrow} (F',T')$;*

(ii) *$A(F,T) \subsetneq A(F',T')$; and*

(iii) *F' does not contain a semi-component of size m_1.*

Proof. Suppose that F does not contain a semi-component of size m_1. Consider an outcome $(\hat{F}, \hat{T})$ such that $\hat{F}_{i,j} = F_{i,j}\mathbb{I}\{i, j \notin A(F,T)\}$ and $\hat{T}_{i,j} = T_{i,j}\mathbb{I}\{i, j \notin A(F,T)\}$.

There are two cases to consider: Either (i) $(\hat{F}, \hat{T})$ contains a semi-component of size m_1 or (ii) the opposite.

If it is case (ii), $(\hat{F}, \hat{T})$ satisfies all three conditions of the lemma. Indeed, the payoff of every agent in $A(F,T)$ in outcome $(\hat{F}, \hat{T})$ is below $r(1,N) < V_1$; hence, $A(F,T) \subset A(\hat{F}, \hat{T})$. By lemma A1.7, there exists at least one agent in $N \setminus A(F,T)$ whose payoff in outcome $(\hat{F}, \hat{T})$ is strictly below V_1; therefore, $A(F,T) \neq A(\hat{F}, \hat{T})$.

Consider case (i), in which $(\hat{F}, \hat{T})$ contains a semi-component of size m_1. Note that $(\hat{F}, \hat{T}) \neq (F,T)$, since (F,T) does not contain a semi-component of size m_1. There are two possibilities: Either $F_{i,j} = 0$ for all $i, j : |\{i,j\} \cap A(F,T)| = 1$ or the opposite. In the latter case, pick a player $k \notin A(F,T)$ such that $\sum_{i \in A(F,T)} F_{i,k} > 0$ and consider an outcome $(F', \hat{T})$, such that $F'_{i,j} = F_{i,j}(\mathbb{I}\{i, j \notin A(F,T)\} + \mathbb{I}\{i = k \text{ and } j \in A(F,T)\} + \mathbb{I}\{j = k \text{ and } i \in A(F,T)\})$. The outcome $(F', \hat{T})$ does not contain a semi-component of size m_1. Moreover, by convexity of R, $A(F', \hat{T}) = N \setminus \{k\} \supset A(F,T)$; hence, $(F', \hat{T})$ satisfies the conditions of the lemma.

Finally, in the former case, when $F_{i,j} = 0$ for all $i, j : |\{i,j\} \cap A(F,T)| = 1$, there exists an agent $k \notin A(F,T)$ such that $\sum_{i \in A(F,T)} T_{k,i} > 0$ and

$$\sum_{i \in A(F,T) \cup \{k\}} U_i(F,T) < (|A(F,T)| + 1)V_1. \tag{A1.3}$$

Indeed, if this condition does not hold, total welfare in (F,T) is above nV_1, which is a contradiction. Moreover, there exists (F, T'') such that for all $i, j \in N \setminus A(F,T) : T_{j,i} = T''_{j,i}$, for all $i \in A(F,T) : T_{i,k} \geq T''_{i,k}$ and for all $i \in A(F,T) \cup \{k\} : U_i(F, T'') < V_1$. Put differently, agents in $A(F,T)$ can reduce their transfers to some agent k to a point at which agent k's payoff drops below V_1. At the same time, payoffs of agents in $A(F,T)$ cannot become larger than V_1 because they can equally distribute their surplus and because their total payoff cannot exceed $|A(F,T)|V_1$ (see (A1.3)). The outcome (F,T'') satisfies the conditions of the lemma: $A(F,T'') = A(F,T) \cup \{k\}$, $(F,T) \overset{A(F,T)}{\to} (F,T'')$, and (F,T'') does not contain a semi-component of size m_1. $\square$

I show that the set $\mathcal{R}$ satisfies internal and external stability. I start with internal stability, and show that for any $(F',T'),(F,T) \in \mathcal{R}$, $(F,T) \not\triangleright (\bar{F},\bar{T})$. Let $\mathcal{H} = \{H_1,...\}$ be a partition that induces (a network of collaboration in) (F,T) and $\mathcal{G} = \{G_1,...\}$ be a partition that induces $(\bar{F},\bar{T})$. Also, let $K = \{i \in N : U_i(F,T) > U_i(\bar{F},\bar{T})\}$. Denote the index of a largest set infiltrated by agents from K in (F,T) by k; i.e., for all $j < k : K \cap H_j = \emptyset$ and $K \cap H_k \neq \emptyset$. Let $M = \bigcup\limits_{j \leq k} G_j$, and note that $|M| > \frac{N}{2}$. For any $S \subset N \setminus M$ and for any $(\hat{F},\hat{T}) : (\bar{F},\bar{T}) \overset{S}{\to} (\hat{F},\hat{T})$, I have $U_M(\bar{F},\bar{T}) = U_M(\hat{F},\hat{T})$. Hence, if $(F,T) \triangleright (\bar{F},\bar{T})$, it must be that $U_M(\bar{F},\bar{T}) = U_M(F,T)$, which contradicts $K \cap M \neq \emptyset$.

To show that $\mathcal{R}$ satisfies external stability, for any $(F',T') \notin \mathcal{R}$ I construct $(F,T) \in \mathcal{R} : (F,T) \triangleright (F',T')$.

I show that for any $(F',T') \notin \mathcal{R}$ that does not contain an isolated group of size m_1, one can always find a network (F,T) such that it contains an isolated group $N_1 : |N_1| = m_1$, $(F',T') \overset{N_1}{\to} (F,T)$ and $(F,T) \triangleright (F',T')$.

I start with the observation that a set $A(F',T') = \{i : U_i((F',T')) < V_1\}$ is nonempty: This follows directly from Lemma A1.7. Suppose $|A((F',T'))| < n/2$. I apply Lemma A1.12 repeatedly to obtain a sequence of outcomes $\{(F^i,T^i)\}$ such that the last element of the sequence, (F^L,T^L), satisfies $|A(F^L,T^L)| \geq n/2$.

There exists (F^{L+1},T^{L+1}), in which all agents in $A(F^L,T^L)$ form a complete component and all but one agent (call him i^*) receive a payoff of zero (this can be achieved by transferring all surplus to i^*). Observe that $A(F^{L+1},T^{L+1}) = N \setminus \{i^*\}$. Select $B \subset N \setminus A(F^L,T^L)$ such that $|B| = m_1 - |A(F^L,T^L)|$ and consider an outcome (F^{L+2},T^{L+2}) in which the agents in B terminate all their relationships to others in $N \setminus A(F^L,T^L)$ and link up with everyone in $A(F^L,T^L)$ but i^*. In addition, all agents who are paying i^* terminate their transfers. Finally, consider an outcome (F^*,T^*) that is obtained from (F^{L+2},T^{L+2}) by adding all missing links between agents in $A(F^L, T^L) \cup B$. This outcome contains a complete component formed by agents in $N_1 = A(F^L, T^L) \cup B$ (there are m_1 of them). As in the proof of Theorem A1.5, the rest of the sequence is constructed using an induction.

□

In the absence of transfers, there are stable outcomes that maximize the total surplus subject to unequal division of tournament prizes. These outcomes are described in Theorem A1.10. Allowing transfers may even be harmful for welfare, as the presence of transfers destabilizes these outcomes. The following result rules out not only these singleton sets, but also any other sets of outcomes that are characterized by a single payoff vector.

Theorem A1.13. *If* $n \notin \arg\max_{\frac{n}{2}<m\leq n}\{r(1,m)+f(m)\}$, *there exists no stable set* $\mathcal{R}$ *such that for any* $(F,T),(F',T') \in \mathcal{R}$, *and for all* $i \in N, U_i(F,T) = U_i(F',T')$.

Proof. I show that any set of outcomes characterized by a single payoff vector necessarily violates external stability.

Take a set $\mathcal{R}$ such that for any $(F,T),(F',T') \in \mathcal{R}$ and for all $i \in N$: $U_i(F,T) = U_i(F',T')$. Without loss of generality, assume that agents are enumerated in such a way that $i > j$ implies that $U_i(F,T) \geq U_j(F,T)$. Consider

$$m_1 \in \arg\max_{\frac{n}{2}<m\leq n}\{r(1,m)+f(m)\}.$$

Note that by conditions of the lemma, $m_1 < n$ and $V_1 = r(1,m_1)+f(m_1) > r(1,n)+f(n)$.

I construct an outcome $(\hat{F},\hat{T})$ such that it is not blocked by any outcome in $\mathcal{R}$. Partition a set $\{1,...,n\}$ into two sets, $N_1 = \{1,...,m_1\}$ and $N_2 = \{m_1+1,...,N\}$ and consider an outcome $(\hat{F},\hat{T})$ such that

(i) $\hat{F}_{i,j} = \mathbb{I}\{\{i,j\} \subset N_1\}$;

(ii) $\hat{F}_{i,j} = 0$ implies $T_{i,j} = 0$; and

(iii) $\forall i \in N_1 : U_i(\hat{F},\hat{T}) > U_i(F,T)$.

There always exists a system of transfers that satisfies condition (iii), because

$$\frac{1}{m_1}\sum_{i=1}^{m_1} U_i(F,T) \leq \frac{1}{n}\sum_{i=1}^{n} U_i(F,T) \leq r(1,n) + f(n)$$
$$< r(1,m_1) + f(m_1) = \frac{1}{m_1}\sum_{i=1}^{m_1} U_i(\hat{F},\hat{T}).$$

By construction, for any $S \subset N_2$ and for all $(F',T') : (\hat{F},\hat{T}) \xrightarrow{S} (F',T')$: $\mathbf{U}_{N_1}(F',T') = \mathbf{U}_{N_1}(\hat{F},\hat{T}) > \mathbf{U}_{N_1}(F,T)$. Therefore, $(\hat{F},\hat{T})$ is not blocked by any outcome that induces the payoff vector $\mathbf{U}(F,T)$. □

A1.2 Relationship to other solution concepts

The most popular solution notion used in the literature on network formation is the pairwise stability introduced by Jackson and Wolinsky (1996). In this section, I discuss the difference between farsighted stable sets and pairwise and setwise stable outcomes in my model.

This section sheds light on one important difference between my model and the models of Goyal and Joshi (2003) and Marinucci and Vergote (2011). In these papers, the fact that links are costly plays an important role in the analysis, as it creates a barrier for firms to create additional links. In this section I combine the solution concept used in these papers with my assumption of beneficial (rather than costly) collaborative links and compare the results with my main findings.

For the rest of the section, I assume that indirect collaboration has no effect on the output. This means that only direct collaboration is beneficial to the agents.

Assumption A1.14. $\alpha_k = 0$ *for all* $k > 1$.

I use the following version of pairwise and setwise stability.

Definition A1.15 (Jackson and Wolinsky, 1996)**.** *An outcome* $(F,0)$ *pairwise blocks an outcome* $(F',0)$ *if either* *(i)* $F - F' = I(\{i,j\})$ *and* $\mathbf{U}_{\{i,j\}}(F,0) \gg \mathbf{U}_{\{i,j\}}(F',0)$ *for some* $i,j \in N$*; or*

(ii) $F' - F = I(\{i,j\})$ *and* $U_i(F,0) > U_i(F',0)$ *for some* $i,j \in N$.

An outcome is pairwise stable if there exists no outcome that pairwise blocks it.

Pairwise stable outcomes are required to be immune to any two agents who create a new link and any single agent who removes one of his existing links. The definition of setwise stability expands the set of permitted deviations by allowing any coalition to create new links between its members and delete any set of links that touch its members at the same time.

Definition A1.16. *An outcome* $(F,0)$ *setwise blocks an outcome* $(F',0)$ *if there exists* $S \subset N$*, such that*

(i) $F_{i,j} > F'_{i,j}$ *implies* $i,j \in S$*;*

(ii) $F_{i,j} < F'_{i,j}$ *implies* $i \in S$ *or* $j \in S$*; and*

(iii) $\mathbf{U}_S(F,0) \gg \mathbf{U}_S(F',0)$.

An outcome is setwise stable if there exists no outcome that setwise blocks it.

In my model, creation of a link between two agents does not have any effect on their relative rankings: Indeed, if one agent ranked higher than the other before the link is created, he still ranks higher once the link is in place. Therefore, creating a link is an immediate improvement for any two agents. This observation easily translates into the following characterization of pairwise stable outcomes.

Remark A1.17. *(i) A unique pairwise stable outcome is the complete network* $(I(N),0)$.

(ii) If for all m *such that* $\frac{n}{2} < m < n$,

$$r(1,n) + f(n) \geq r(1,m) + f(n-2) + [f(n-1) - f(n-2)]\mathbb{I}\left\{m \geq \frac{2n}{3}\right\},$$

the complete network $(I(N), 0)$ *is a unique setwise stable outcome; otherwise, there exists no setwise stable outcome.*

Proof. Consider an arbitrary outcome $(F, 0)$. A pair of agents $i, j \in \{1, ..., N\}$ pairwise blocks $(F, 0)$ if and only if $F_{i,j} = 0$. Therefore, a unique pairwise unblocked outcome is the one that features a complete network of collaboration.

Note that the pairwise blocking relation is a subset of the setwise blocking relation. Therefore a setwise stable outcome exists iff a pairwise stable outcome is not setwise blocked. Consider a coalition $S : |S| = m$ currently residing in outcome $(I(N), 0)$. This coalition can obtain the top rankings for its members by severing either one or two links per member of the coalition. Indeed, if $m \geq 2n/3$, one link per member would suffice; otherwise, two links are necessary. The resulting payoff for a member of coalition S is

$$r(1, m) + f(n-2) + [f(n-1) - f(n-2)]\mathbb{I}\left\{m \geq \frac{2n}{3}\right\}.$$

The complete network is not blocked by another outcome that results from such a deviation if the above payoff is below $r(1, n) + f(n)$.

The coalitional deviations described above are the best (payoff-wise) in the class of deviations that ensure equality among members of the deviating coalition. If members of S are not better off as a result of these deviations, at least one of them is not better off as a result of any other deviation, because R is convex. □

Two important assumptions are required for this result. The first is that the agents are minimally coordinated—i.e., coalitions of three or more players cannot coordinate their actions (this guarantees existence in part 1 of Remark A1.17). The second, more important assumption is that agents ignore the effect of their actions on other agents' incentives. When these assumptions are replaced by agents' farsightedness and their ability to coordinate, the uniqueness and efficiency of the stable outcome presented in Remark A1.17 is replaced by a negative result in Theorem A1.9. The findings of Dutta et al. (1998) have a similar flavor; however, their result holds only in three-player majority games.

Remark A1.17 shows that unless links are costly, there is no tension between efficiency and pairwise stability. The stable networks in Theorems A1.5 and A1.11 look similar to the ones discussed in the theoretical literature on collaboration, but these networks arise under conditions that are regarded in the literature as favorable for efficient outcomes.

The externality that agents impose on each other is to blame for the inefficiency of outcomes in a farsighted stable set. In general, this does not mean that farsighted stability is more prone to selecting inefficient outcomes than pairwise stability. In the coauthorship model discussed in Jackson and Wolinsky (1996), the opposite is true. Pairwise stable outcomes are inefficient due to the negative externality agents who are indirectly connected impose on each other, and farsighted stable outcomes are always efficient because a pair of agents can always leave their coauthors and work together exclusively. The nature of externalities is different in these two models, and when it is backed by asymmetry in the rules for creating and deleting links, it results in qualitatively different predictions for both solution concepts.

A1.3 Consistency of stable sets

The logical construction of a stable set is further reinforced by the consistency property. Consistency of a set of outcomes means that any profitable deviation from an outcome in this set is followed by a path back into the set; moreover, the path back is such that one of the original deviators' payoffs is below the pre-deviation level.

Chwe (1994) shows that a farsighted stable set possesses the consistency property. In the original proposition, Chwe (1994) formulates this property for one-step deviations, but it can easily be extended to sequential deviations.

Remark A1.18 (Chwe, 1994)**.** *Let $\mathcal{R}$ be a farsighted stable set. Take $(X,T) \in \mathcal{R}$ and any $(X',T') \triangleright (X,T)$. For any sequence underlying this blocking $\{(S_k, X^k, T^k)\}_{k=1}^{K}$ and for any outcome $(X'',T'') \in \mathcal{R}$ such that $(X'',T'') \triangleright (X',T')$, there exists* *an agent* $i \in \bigcup_{k=1}^{K} S_k$ *such that*

$$U_i(X,T) \geq U_i(X'',T'').$$

Proof. By contradiction, suppose that for all $i \in \bigcup_{k=1}^{K} S_k$,

$$U_i(X,T) < U_i(X'',T'').$$

Note that $(X,T) = (X^1,T^1) \overset{S_1}{\to} (X^2,T^2) \overset{S_2}{\to} \dots \overset{S_K}{\to} (X',T')$ implies that $(X,T) \overset{S}{\to} (X',T')$ where $S = \bigcup_{k=1}^{K} S_k$. Then $(X'',T'') \triangleright (X,T)$, which contradicts the internal stability for $\mathcal{R}$. □

Intuitively, this property of farsighted stable sets means that there is a punishment for any profitable deviation from a stable outcome. Any deviation from a stable outcome ultimately results in a transition back to another outcome in a stable set, and there is always at least one player among the original deviators who is worse off.

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