Consumer storage confronts monopoly power

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We study the steady state behavior of the market for a storable good where firms have monopoly power instantaneously, but compete against future sellers. Consumers have identical preferences, but differ in their willingness to pay due to differential inventory holdings. In a steady state, the optimal nonlinear tariffs chosen by the firms induce the constant distribution of private inventories. Identical consumers behave differently, shop infrequently and consume in a cyclical manner. The ability to store goods gives rise to inefficiency, but also allows consumers to retain some surplus.

Keywords: storable good, dynamic nonlinear pricing.

1 INTRODUCTION

Many goods are storable, which means that the consumer can buy a large quantity and consume gradually from her inventory. Furthermore, the consumer's inventory is unobservable by the seller, which puts significant restrictions on the seller's ability to exercise monopoly power. This point is made most clearly by Hendel, Lizzeri and Roketskiy (2014), who study a monopolist who faces a population of homogeneous consumers, with identical preferences. The monopolist can engage in unrestricted nonlinear pricing. They show that the consumer's ability to store and to purchase only occasionally puts significant restrictions on monopoly power. This leads the monopolist, who is assumed to be able to commit to the future path of prices, to set prohibitively high "normal" prices, interspersed by periodic sales, which are the only periods in which the consumer buys. Nonetheless, casual empiricism suggests that many storable goods such as soft drinks, bottled water or toiletries are not characterized by a combination of prohibitive prices and periodic sales, and aggregate demand is relatively stationary in these markets.[1]

The present paper differs from Hendel et al. (2014) in several respects. First, we assume that the firm's monopoly power is transitory, as in a dynamic Diamond search model. That is, the firm is effectively the exclusive seller to the consumer who visits it in the current period, but it recognizes that the consumer has the option of postponing purchase to the future, in which case she will buy from a different firm. Second, although consumers have identical preferences (as in Hendel et al., 2014), they will have a distribution of demands as a consequences of differential inventories, and the firm uses nonlinear prices as a way of screening different inventory levels. Finally, the resulting equilibrium is stationary—in each period, the distribution of consumer inventories is constant, leading each "transitory monopolist" to offer a constant nonlinear tariff.

We show that menu pricing of storable goods is very different from the standard (static) second-degree price discrimination. Our model predicts the following:

(i) Identical consumers become endogenously heterogeneous in the amount of good they keep in their inventory.

[1]It should be noted that even with full commitment, the monopolist's optimization problem is complex due to the storage, and the paper cannot characterize the fully optimal dynamic nonlinear price schedule.

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(ii) Sellers respond to consumer heterogeneity by offering a variety of pack sizes. A clear divide arises between bundles designed for immediate consumption and bundles designed for stockpiling.
(iii) Consumers shop and consume in a cyclical manner. They visit shops infrequently and rely on their inventories for consumption. The frequency of shopping is determined by how inefficient consumer storage is.
(iv) Despite nonlinear pricing, the consumers retain surplus which does not depend on how difficult or costly it is to store the good.
(v) Rather than reflecting a persistent taste heterogeneity, equilibrium menus of pack sizes are shaped by the intertemporal elasticity of demand.
(vi) Sellers make the good more scarce when better information about past prices is available.
(vii) Sellers who have long-lasting monopoly power make the good more scarce than sellers who serve the consumers only occasionally.

In the absence of storage, nonlinear pricing with homogeneous consumers is straightforward. It suffices for the seller to sell a single quantity in each period, the efficient one, at a price which extracts all the consumer's surplus. This is not possible when the good is storable, since the consumer can obtain the good required for the current consumption from two sources: she can buy it from the seller, or she could reduce her past consumption and transfer the good to the current period using storage. The latter is equivalent to buying the good at a linear (shadow) price that is proportional to the past marginal utility of consumption. To be competitive, the seller is forced to lower the price and offer more than one bundle on the menu. This occurs not once, but at each moment in time. In equilibrium, it makes it cheaper for the consumer to source the good at all times, and therefore puts even more pressure on the seller. This mechanism is at the heart of our results.

In this paper, we focus on stationary equilibria. We do so both for tractability and for realism, since we believe that for many storable goods, prices are relatively stable. Of course, there are other goods which are characterized by alternating periods of high and low prices. This is indeed possible if monopoly power is persistent, the seller faces little competition from other sellers even in the future, and has the ability to commit to future prices , as in Hendel et al. (2014). In our setting, there could possibly be non-stationary equilibria, but characterizing them seems intractable, since even the stationary equilibrium is relatively complex.

In the next section we discuss the related literature. We set up the model in Section 3, discuss key equilibrium conditions in Section 4 and characterize stationary equilibria in Section 5. Section 6 studies the effects of information and market structure on equilibrium pricing and Section 7 concludes.

2 RELATED LITERATURE

Nonlinear pricing of storable goods has been studied by Hendel et al. (2014). They consider a setting in which the good is sold by a monopolist that is able to commits to future prices. The monopolist expects the consumers to stockpile in periods of low prices and to avoid shopping in periods of high prices. The monopolist can limit intertemporal arbitrage by setting a high per-unit price and running periodic sales during which consumers get a chance to stockpile.

The main difference in our analysis is that we allow for competition between the current period sellers and future sellers. This competition eliminates the seller's incentive to reduce consumer heterogeneity via differential inventories. Consequently, our model's predictions are different from Hendel et al. (2014). We show that stationary equilibrium is possible. In this equilibrium, homogeneous consumers diverge in their choices of inventories. Instead of using periodic sales, the sellers rely on the menu design to extract rent from consumers. They introduce two distinct type of bundles meant for either immediate consumption or stockpiling.

One common theme that arises in our analysis and in Hendel et al. (2014) is that consumer surplus does not depend on the details of storage technology. They show that the storage capacity[2] is irrelevant for the seller's profit and consumer surplus because consumers can adjust the frequency of shopping to offset the limitations of their storage.

The rest of the literature on storable goods focuses either on linear pricing or on models with unit demand. The effect of sellers' commitment to future prices is studied by Dudine, Hendel and Lizzeri (2006). They show that both consumers and the monopolist benefit from the latter's ability to credibly set future prices. Berbeglia, Rayaprolu and Vetta (2019) point out that this this result crucially depends on divisibility of the good. They study a version of the model with a unit demand and identify cases in which a monopolist benefits from contingent pricing.

The effects of consumer inventories are well-understood when the seller is limited to using linear pricing. Anton and Das Varma (2005) show that competing sellers temporarily lower prices in an attempt to capture future rivals' market shares (also, see Guo and Villas-Boas, 2007). Other studies of price dynamics for storable goods include Benabou (1989), Deaton and Laroque (1996), Su (2007), Su (2010) and Antoniou and Fiocco (2024).

Antoniou and Fiocco (2019) consider the strategic use of inventories by the sellers and Antoniou and Fiocco (2023) focus on variability of production costs. Nava and Schiraldi (2014) show that consumer inventories can be used as a vehicle to sustain sellers' collusion.

The evolution of the consumer inventories is driven by the past consumption and shopping decisions, therefore, consumer histories are relevant for pricing and other related issues such as the efficient allocation of resources. Cole and Kocherlakota (2001) study the social planner's problem of allocating a storable resource in the context of hidden savings. Their work focuses on optimal insurance rather than surplus extraction. They show that the optimal allocation is equivalent to a market outcome in which consumers can trade risk-free bonds.

Bhaskar and Roketskiy (2021) investigate the importance of privacy for markets in the history of past consumption is relevant for current pricing. Our model shares a set of features with these two studies. Similar to savings, inventories are private, they affect the consumers' current valuation for the good and they are shaped by the past shopping and consumption decisions.

[2] One can view storage capacity as a manifestation of a particular type of storage cost for which the first S units of good are stored for free and anything on top of S is stored at a prohibitively high cost.

Inventories affect the value consumers' outside options and as a result may cause price dispersion as was shown in Hong, McAfee and Nayyar (2002). Moreover, uneven change in inventories may cause dynamic changes in aggregate demand. The interaction between a dynamic demand and pricing has been studied in the context of durable good monopolist (e.g., by Sobel (1991) and Garrett (2016)). Economic literature draws a clear distinction between durable and storable goods. Durable goods are characterized by unit demand, exogenous depreciation and passive consumption when the consumer decides only when to purchase the good (or to replace it). Storable goods, on the other hand, are usually assumed to be homogeneous and perfectly divisible. In addition to deciding when and how much of the good to purchase, the consumers choose the rate of consumption. In practice, these distinctions are less pronounced and there are goods which exhibit the traits of both theoretical categories.

A key component of our analysis is optimality of nonlinear pricing in a dynamic setting. This problem is related to the problem of optimal durable good pricing and Coase conjecture (see Coase, 1972). A standard assumption in the literature on durable good pricing is that the consumers leave the market after purchasing the good. This assumption delivers a relatively simple characterization of the distribution of the willingness to pay for the consumers remaining in the market—usually it is a truncation of the initial distribution (see Doval and Skreta, 2019).

The property that sets storable goods apart from durable ones is frequent repeated shopping: consumers return to the market periodically to top up their inventories. Moreover, their current demand is determined by their past consumption decisions. This makes the problem of characterizing aggregate demand complicated. In addition to an adverse selection aspect which is common for durable good pricing problems, our model has a moral hazard component. Past studies of dynamic moral hazard and adverse selection include among others Ma (1991), Strulovici (2011), Williams (2015), Halac, Kartik and Liu (2016) and Bhaskar and Roketskiy (2023).

On the empirical front, Hendel and Nevo (2004) reviews early literature on markets for storable goods. Hendel and Nevo (2006a,b) estimate elasticity of demand for storable good (laundry detergents, soft drinks and yogurts) and point out that elasticity estimated using a static demand is biased upward by a large factor.[3] Therefore, consumer stockpiling behavior is an important consideration for pricing. Our model includes the supply side's response to consumers' strategic stockpiling—an element that is not accounted for in Hendel and Nevo's analysis.

Hendel and Nevo (2013) use a model with linear pricing and seller's commitment to future prices to measure the effect of inventory-related intertemporal price discrimination on profits and welfare. Even though our model focuses on stationary equilibria, it is clear from our results that there is a scope for cyclical pricing patterns even when sellers cannot credibly set prices for the future periods.

Estimating elasticity is one of many tasks for which the dynamic nature of demand for storable goods has to be taken into account. Another one is aggregating prices into

[3]Kano (2018) presents survey evidence of the use of inventories and its effect on consumer demand. Ching and Osborne (2020) discusses identification of consumer stockpiling behavior.

macroeconomic price indices. Osborne (2018) proposes a method of calculating cost-of-living index that accounts for the use of inventories in intertemporal arbitrage.

3 THE MODEL

A unit measure of identical and anonymous consumers comprises the market for a perfectly divisible and storable good. Each consumer is long-lived and has quasi-linear preferences over the stream of consumption and expenditures $\{(c_t, p_t)\}_{t=0,1,\ldots}$. These preferences are represented by a utility function

$$\sum_{t=0}^{\infty} \delta^t [u(c_t) - p_t],$$

where u is an increasing, concave and infinitely differentiable function that represents the flow value of consumption. p_t is the total expenditure on the good in period t (it is not the unit price, since we allow nonlinear pricing). We assume that $u(0) = 0$, $\lim_{c\to 0} u'(c) = \infty$ and $u'''(c) \leq 0$ for all c.

A consumer can store the good at a cost that is measured in units of consumption good.[4] We denote the level of consumer i's inventory at date t by s_t. In order to have this amount of good in storage, the consumer must put $h(s_t)$ units of the good into the storage at date $t-1$. We assume that h is increasing, convex and infinitely differentiable. The function h represents the (gross) cost of storing the good, therefore $h'(0) \geq 1$.

Let x_t be the amount of good purchased by consumer i at time t. Consumption and storage choices are feasible if they satisfy the following resource constraint:

$$x_t + s_t = c_t + h(s_{t+1}).$$

We model the supply side of the market as a serial monopoly. In each period t, there is a single short-lived seller also called t, who can procure any quantity of the good at marginal cost k and sell it to the consumers.[5] Each seller can use nonlinear menu pricing, but cannot offer long-term contracts, and therefore, must transact with consumers on the spot. Formally, a menu is a lower semi-continuous function $p_t(x)$ that assigns a nonnegative price to each quantity x. Any menu includes the option of not buying anything—i.e., $p_t(0) = 0$. Sellers set their quantity-price menus to maximize profit which equals to the total revenue net of procurement costs.

Even though the consumers are identical in terms of their preferences and constraints, they may consume and store differently from each other. Thus, in any period, consumption, storage and expenditures are consumer specific, and should be indexed by i, the consumer's identity. The aggregate profit of seller t is a sum of profits from individual transactions with each consumer i:

$$\int_0^1 (p_t(x_{i,t}) - kx_{i,t})di.$$

[4]Modelling this cost in monetary units would produce qualitatively similar results.

[5]There need not literally be a single seller in each period. One could have a large number of identical sellers, with the consumer having to pay a search cost to visit a second seller, as in Diamond (1971).

3.1 Timing, information and equilibrium notion

A consumer's past decisions—i.e., how much to purchase, store and consume—are private and not observed by the seller at date t. Nor does the seller t observe the menus that were offered to the consumers by the his predecessors.

In the beginning of period t, consumer i chooses whether to visit seller t or not. If he visits the seller, she observes the menu and decides what item to buy. If the consumer chooses not to visit seller t, she cannot buy anything in that period.[6]

In Section 6 we use a simplified, two-period model to examine how our findings depend on information and market structure: we consider scenarios in which menus are publicly observable and a single long-lived seller replaces serial monopoly.

4 KEY EQUILIBRIUM CONDITIONS

First, we establish a general relationship between the consumption rule and the menus offered by the sellers throughout time. We do it without imposing any additional assumptions—such as stationarity—on equilibrium. We present two conditions: one characterizes optimal consumer's choice from the sellers' menus, and the other characterizes optimal allocation of the good across different periods.

The first condition takes a form of an integral inequality that in some special cases (but not always) reduces to a more familiar condition—the monotonicity of the consumers choice from the menu in her inventory level. This relationship is presented in Proposition 1. The second condition is an Euler equation. It characterizes how consumers allocate the good in their inventory across time. This condition appears in Proposition 2

After this we turn to the optimality of prices. We describe how consumers with nonempty storage discipline sellers' ability to extract surplus by restricting consumers' choice in equilibrium. Proposition 3 presents the formal argument.

Consider a consumer with inventory s at time t. Let $V_t(s)$ be a value function for this consumer prior to buying from the seller. Then,

$$V_t(s) = \max_{c,\hat{s},x}\{u(c) + \delta V_{t+1}(\hat{s}) - p_t(x)\} \tag{1}$$
$$\text{s.t.} \quad x + s = c + h(\hat{s}).$$

Similarly, let $W_t(r)$ be the consumer's value after she has visited the seller. Variable r represents the amount of the good at consumer's disposal. Then,

$$W_t(r) = \max_{c,\hat{s}}\{u(c) + \delta V_{t+1}(\hat{s})\} \tag{2}$$
$$\text{s.t.} \quad r = c + h(\hat{s}).$$

Note that the consumer's optimization can be done in two steps: first choosing the best item from the menu $x_t(s)$ given the current inventory s and then choosing the consumption amount $\tilde{c}_t(r)$ given the total resources r (which include the purchased item). This follows from that fact that consumers who have the same inventory obtained in different ways

[6]This assumption, that the consumer cannot buy unless he visits the seller, simplifies the analysis. We discuss this assumption in further detail in Section 5.2

face the same allocation problem. Thus, we can replace (1) and (2) with

$$V_t(s) = \max_{x \geq 0}\{W_t(s+x) - p_t(x)\}$$
$$W_t(r) = \max_{\hat{s} \in [0, h^{-1}(r)]}\{u(r - h(\hat{s})) + \delta V_{t+1}(\hat{s})\}$$

In contrast with a static optimal menu pricing, the complication that arises in a dynamic environment is that the optimality of the consumption rule cannot be formulated for a single isolated period. Optimal consumption depends on the current and the *anticipated future prices.* Consequently, the two value functions, V and W, depend upon the sequence of anticipated prices.

Proposition 1. *Consider a solution $\tilde{c}_t(\cdot)$ to a programme* (2). *Consumer's choice of an item from the menu $x_t(s)$ (and corresponding consumption $c_t(s) = \tilde{c}_t(x_t(s) + s)$) is optimal given some menu $p_t(x)$ if and only if*

(i) value functions are differentiable and satisfy the envelope condition; namely, for any $r \geq 0$:

$$W_t'(r) = u'(\tilde{c}_t(r)), \tag{3}$$

and for any $s \geq 0$:

$$V_t'(s) = u'(c_t(s)); \tag{4}$$

(ii) consumer's choice $x_t(s)$ satisfies

$$\forall s, z \geq 0 : \int_z^s u'\left(\tilde{c}_t(r + x_t(r))\right) dr \geq \int_{z+x_t(z)}^{s+x_t(z)} u'(\tilde{c}_t(r)) dr. \tag{5}$$

Proof. First, consider any $\epsilon > 0$ and r_1, r_2, such that $r_1 > r_2$ and $|r_1 - r_2| < u^{-1}(\epsilon)$. The following

$$|W_t(r_1) - W_t(r_2)| \leq \max_{\hat{s}}\{|u(r_1 - h(\hat{s})) - u(r_2 - h(\hat{s}))|\} = u(r_1 - r_2) < \epsilon$$

together with concavity of u implies that W_t and V_t are absolutely continuous and therefore the envelope theorem applies:

$$W_t(r_2) = W_t(r_1) + \int_{r_1}^{r_2} u'(\tilde{c}_t(r)) dr, \text{ and}$$

$$V_t(s_2) = V_t(s_1) + \int_{s_1}^{s_2} u'(c_t(s)) ds.$$

To see that the value functions are differentiable, consider the following:

$$\begin{aligned}
&V_t(s+\epsilon) - V_t(s) = \\
&\max_{x\geq 0}\{W_t(s+\epsilon+x) - p_t(x)\} - \max_{x\geq 0}\{W_t(s+x) - p_t(x)\} \geq \\
&W_t(s+\epsilon+x_t(s)) - W_t(s+\hat{x}_t(s)) \geq \\
&\max_{h(\hat{s})\leq s+\epsilon+x_t(s)}\{u(s+\epsilon+x_t(s)-h(\hat{s})) + \delta V_{t+1}(\hat{s})\} - \max_{h(\hat{s})\leq s+x_t(s)}\{u(s+x_t(s)-h(\hat{s})) + \delta V_{t+1}(\hat{s})\} \geq \\
&u(c_t(s)+\epsilon) - u(c_t(s)).
\end{aligned}$$

At the limit, when $\epsilon > 0$ vanishes, we get

$$\partial_+ V_t(s) \geq \partial_+ W_t(s+x_t(s)) \geq u'(c_t(s)).$$

Similarly,

$$\partial_- V_t(s) \leq \partial_- W_t(s+x_t(s)) \leq u'(c_t(s)).$$

Since the inventory is a consumer's choice, and $u(\cdot)$ is differentiable, it must be that

$$\partial_- V_t(s) \geq \partial_+ V_t(s).$$

Thus,

$$\begin{aligned}
\partial_- V_t(s) &= \partial_+ V_t(s) \\
\partial_- W_t(s+x_t(s)) &= \partial_+ W_t(s+x_t(s)),
\end{aligned}$$

and both value functions are differentiable at s and $s+x_t(s)$ respectively.

The consumer's choice $x_t(s)$ is optimal if and only if[7]

$$\forall s,z: W_t(s+x_t(s)) - p(x_t(s)) \geq W_t(s+x_t(z)) - p(x_t(z))$$

We can rewrite this inequality as

$$\forall s,z: W_t(s+x_t(s)) + W_t(z+x_t(z)) - p(x_t(s)) \geq W_t(s+x_t(z)) + W_t(z+x_t(z)) - p(x_t(z))$$

or

$$\forall s,z: V_t(s) - V_t(z) \geq W_t(s+x_t(z)) - W_t(z+x_t(z))$$

Using the envelope condition on both sides of the inequality we get

$$\forall s,z \geq 0: \int_z^s u'\left(\tilde{c}_t(r+x_t(r))\right) dr \geq \int_{z+x_t(z)}^{s+x_t(z)} u'(\tilde{c}_t(r))dr.$$

□

In this proposition, the inequality (5) plays a similar role to monotonicity of the allocation with respect to type in static nonlinear pricing models.[8] In Section 5.1, where we construct

[7] In addition to that, one needs to ensure that items on the menu that are never chosen on equilibrium path have prohibitively high prices.

[8] Ours is a model that combines adverse selection with moral hazard. In these models, a double deviations—i.e., agent choosing both the "wrong contract" x and the "wrong effort" s at the same time—potentially presents an additional complication. See Castro-Pires, Chade and Swinkels (2024) for in-depth discussion of the issue. However, in our setting a contract x and "effort" s are additive—this makes double deviations straightforward to deal with.

the equilibrium consumption rule, we verify that the induced $\tilde{c}_t(r)$ indeed satisfies condition (5). After the consumption rule is constructed, we can use envelope conditions (3) and (4) to find the prices under which this rule is consumer-optimal. So far, we focused on the optimality of consumer choice from a single menu. The next result brings intertemporal incentives into the picture. It provides a necessary condition for optimal allocation of the good across time. When combined with the resource constraint and the optimal consumption rule, it pins down the equilibrium behavior of the consumers.

Proposition 2. *Consider a consumer with an inventory s_t in period t. Define the consumer's choice of inventory for the next period as*

$$s_{t+1} = h^{-1}(s_t + x_t(s_t) - c_t(s_t)).$$

If the consumer's choices are optimal then the following inequalities hold and at least one of them binds

$$\begin{aligned} &\delta^{-1} h'(s_{t+1}) u'(c_t(s_t)) \geq u'(c_{t+1}(s_{t+1})) \\ &s_t + x_t(s_t) \geq c_t(s_t). \end{aligned}$$

Proof. Consider an auxiliary problem in which instead of buying $x_t(s_t)$ and $x_{t+1}(s_{t+1})$ from the corresponding menus, the consumer receives these items for free. The optimality of choice of consumption $c_t(s_t)$ and $c_{t+1}(s_{t+1})$ in the original problem implies that the consumer would optimally choose the same consumption levels in the auxiliary problem. The necessary condition for the optimality of consumption in the auxiliary problem is the following:

$$h'(s_{t+1}) u'(c_t(s_t)) = \delta u'(c_{t+1}(s_{t+1})) + \lambda,$$

where $\lambda \geq 0$ is a Lagrange multiplier for the constraint $s_{t+1} \geq 0$. The complementary slackness condition is $\lambda s_{t+1} = 0$. □

Propositions 1 and 2 do not invoke sellers' incentives. Without assuming stationarity of equilibrium, the sellers' profit maximization problem is unwieldy because the state—the distribution of consumers' inventories—is infinite-dimensional and its evolution is complex. Nevertheless, prior to assuming stationarity of this distribution and characterizing the equilibria in Section 5, we can still derive some general properties of equilibria.

A combination of buyer's and seller's sequential rationality drastically narrows down the set of menus that can potentially appear in equilibrium. On the one hand, the seller sets prices in such a way as to leave no surplus to the consumer with certainty (the only surplus that the consumer can retain arises from her private information). On the other hand, the consumer controls what the seller knows with certainty about her valuation through her choice of inventory. The following proposition formalizes this observation.[9]

Proposition 3. *Fix the menu offered by the seller t and consider a consumer who chose to arrive to period t with inventory $s > 0$. Suppose that this consumer purchases y_1 from seller t.*

[9]In nonlinear pricing context, a similar result arises in other settings with a dynamic demand (for example, see Lemma 18 in Bhaskar and Roketskiy, 2021). More generally, it is related to differentiability of agent's value function in models of hold-up (e.g., see Fudenberg and Tirole, 1990, González, 2004, Gul, 2001).

If there is an item $y_2 < y_1$ on the same menu, such that the consumer is indifferent between buying y_1 and y_2. Then $\tilde{c}_t(s + y_1) = \tilde{c}_t(s + y_2)$.

Proof. Let $\tilde{V}(s) = \max_{y \in \{y_1, y_2\}} \{W_t(s + y) - p_t(y)\}$. Because the choice of s is optimal, for any $\epsilon > 0$:

$$u(r_{t-1} - h(s)) + \delta\tilde{V}(s) \geq u(r_{t-1} - h(s - \epsilon)) + \delta\tilde{V}(s - \epsilon)$$
$$u(r_{t-1} - h(s)) + \delta\tilde{V}(s) \geq u(r_{t-1} - h(s + \epsilon)) + \delta\tilde{V}(s + \epsilon)$$

Given the definition of $\tilde{V}$, these inequalities imply that for any $y \in \{y_1, y_2\}$

$$\delta^{-1}[u(r_{t-1} - h(s - \epsilon)) - u(r_{t-1} - h(s))] \leq W_t(s + y) - W_t(s - \epsilon + y)$$
$$\delta^{-1}[u(r_{t-1} - h(s)) - u(r_{t-1} - h(s + \epsilon))] \geq W_t(s + \epsilon + y) - W_t(s + y).$$

Because $u(\cdot)$, $h(\cdot)$, and $W_t(\cdot)$ are differentiable, and $W_t'(r) = u'(\tilde{c}_t(r))$, we can divide both sides of these two inequalities by ϵ, and in the limit, obtain

$$u'(\tilde{c}_t(s + y_1)) \geq u'(\tilde{c}_t(s + y_2)) \geq u'(\tilde{c}_t(s + y_1)),$$

and therefore,

$$\tilde{c}_t(s + y_1) = \tilde{c}_t(s + y_2).$$

□

This proposition can be illustrated with a clear and simple intuition. Even though consumers are limited to shopping with a single seller each period, this seller has to compete in marginal prices with the consumers' past selves. Indeed, instead of purchasing a marginal unit from the seller today, a consumer could consume less in the previous period and increase her inventory. Because the consumers trade with their past selves using linear prices, which are equal to marginal values of consumption, the sellers must do so as well.

This competition à la Bertrand prevents the sellers from extracting the consumer surplus in bulk by setting the prices to equalize the net values of two different options on the menu. The only exception to this rule is a consumer without inventory (i.e., $s = 0$). The seller can sell a high quantity-low marginal price bundle and extract surplus at bulk by asking to pay a fee for an access to this bundle. Despite the fact that the induced consumption in the current period is high, the consumer cannot reallocate it to the previous period by storing less because the inventory is at the zero bound.

This observation has an important implication for the nature of aggregate demand. If there is a "lump" in demand, namely if there is a large number of consumers with the same willingness to pay, the seller's optimal response is a menu that extracts surplus in bulk—a type of a menu that is essentially ruled out by Proposition 3. Thus, in equilibrium, one should expect the market to be composed of (endogenously) heterogeneous population of consumers.

Proposition 4. *In any period except the first, consumers are heterogeneous in inventories and sellers offer a variety of pack sizes on their menus.*

Proof. Recall that every menu must contain the outside option: $p_t(0) = 0$. If there is a seller that does not offer a variety of pack sizes—or more precisely, if there is a menu p_t such that only the single pack size x is purchased from it—there must be a consumer who is indifferent between buying x and not buying anything. Proposition 3 states that either this consumer arrives to seller t without inventory or the consumer's inventory is $s > 0$, but the consumption does not change as a result of a purchase, namely $\tilde{c}_t(s) = \tilde{c}_t(s + x)$. Moreover, because there is only one item for sale and the seller is maximizing profit, it must be the case that the consumption is at the efficient level: $u'(\tilde{c}_t(s + x)) = k$.

If the consumers has an empty inventory, it must be the case that

$$\lim_{c \to 0} u'(c) \leq \delta^{-1} h'(0) u'(c_{t-1})$$

which violates our assumptions for any $c_{t-1} > 0$.

If $s > 0$, because $u'(\tilde{c}_t(s)) = u'(\tilde{c}_t(s+x)) = k$, Proposition 2 implies that marginal utility of consumption is strictly below k in the previous period. This can only happen if one of the previous sellers sold the good at a marginal price strictly below k, which is impossible if this seller is maximizing profit. Thus, in equilibrium in each period except possibly the first one, the seller offer a variety of packs on the menu.

If there is a period t in which consumers are homogeneous in their inventories, it is optimal for the seller t to offer a single pack size, which is a contradiction. □

The key to this result is Proposition 3. On the one hand, a single item in the menu combined with the seller's sequential rationality implies that the surplus is extracted in bulk in a nonlinear fashion. On the other hand, Proposition 3 implies that everything that is sold in this period is stored. Put differently, the consumer resells what she purchased to her future self using a linear (shadow) price which is equal to the marginal utility of consumption. If a good is bought at a nonlinear price and resold at the linear one, the consumer incurs a loss, and therefore, she would better off not doing it.

Propositions 1, 2 and 3 combined with the resource constraints are sufficient to characterize the consumers' behavior and prices in stationary equilibria. The results in Propositions 1, 2, 3 and 4 do not rely on the fact that the sellers are short-lived or that they do not observe the menus offered to consumers in the past. When we study long-lived monopolist in Section 6, we use these results to characterize the equilibrium prices for that alternative setting.

5 STATIONARY EQUILIBRIA

In a stationary equilibrium[10], any two sellers face the same aggregate demand for their good; the menus and consumers' value functions do not depend on calendar time. To simplify exposition and pave the way for clear intuition behind our results, we assume that the cost of storage is linear. Generalizing our analysis to a case of nonlinear storage cost is straightforward.

Assumption 5. $h(s) = \beta s,\ \beta > 1$.

[10]Formally, to define a stationary equilibrium we either need to make the time horizon infinite in both directions, or assume an initial cross-section distribution of inventories that coincides with the stationary one.

In the equilibrium that we construct, consumers' choices follow a cyclical pattern. The cycle starts when the consumer attends a shop and buys a large amount of good to fill her inventory. After that, the consumer does not shop for several periods and consumes the stored good. This gradually depletes her inventory. Towards the end of the cycle, there is insufficient amount of stored good to fulfill her consumption requirement. At that point, the consumer visits the store and purchases a small top-up bundle. Then, she consumes the bundle together with leftovers to arrive to the beginning of the next cycle with the empty storage.

In addition, when stockpiling, different consumers purchase different amounts, thus creating a heterogeneity in inventories. This heterogeneity persists throughout the cycle. The sellers respond to it by offering screening menus. In the remainder of this section we formalize these ideas and discuss every aspect of this equilibrium in great detail, starting with consumption.

5.1 Equilibrium consumption

The condition for optimality ties the levels of consumption across adjacent periods together. This allows us to construct the consumption stream and then find the (nonlinear) prices that, on the one hand, maximize seller's payoff and, on the other, induce this consumption stream.

We conjecture and verify that every consumer follows the same consumption sequence in the stationary equilibrium (in an asynchronous way).[11] Consider a consumer who has some amount of good stored: $s_t > 0$. Proposition 2 implies that

$$\beta u'(c_{t-1}) = \delta u'(c_t).$$

Because the sellers are short-lived, a standard no-distortion-at-the-top argument implies that a consumer with no inventory must consume at the efficient level $c^* : u'(c^*) = k$. Thus, we can define a sequence $\{c^\tau\}_{\tau=0,1,\ldots}$ that satisfies

$$u'(c^\tau) = (\delta^{-1}\beta)^\tau u(c^*) = (\delta^{-1}\beta)^\tau k. \tag{6}$$

For cyclical behavior, it is convenient to use an index that indicates the time period withing a cycle. This index τ runs from 0 to some T and then gets reset to 0 to start a new cycle.[12] The length of the cycle $T + 1$ is endogenous and it depends on the storage technology β. Intuitively, the better the storage technology, the less frequent the shopping, the longer the cycle.

Let $\hat{c}(s), \hat{x}(s), \hat{s}(s)$ and $\hat{a}(s)$ be stationary consumer-optimal rules that describe consumer choices conditional on the current inventory level s. As shown on Figure 1, at the beginning of each cycle, consumer's storage is empty. The consumer buys a large amount of good $\hat{x}(0) = x^0$ to fill up her storage. She consumes the efficient amount $\hat{c}(0) = c^0 = c^*$ and puts the rest of the good into the storage:

$$\beta\hat{s}(0) = x^0 - c^0.$$

[11]This is a consequence of linear costs of storage. Strictly convex costs would cause heterogeneity in consumption streams. Similarly, a heterogeneity in storage cost would also imply variation in consumption streams across consumers.

[12]We denote calendar time with a subscript and time within a cycle by a superscript.

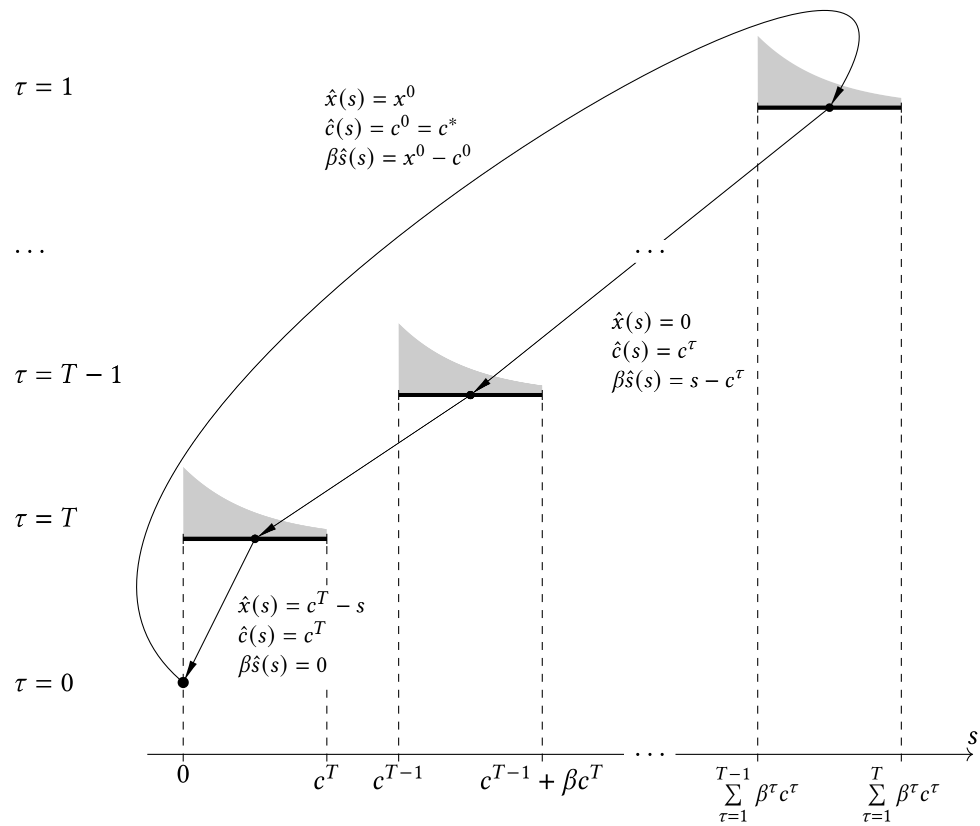


Fig. 1. Consumer cycles

Note: A diagram of a consumer cycle with transitions illustrated for a single consumer who purchases $\hat{x}(0) = x^0 \in \left[\sum_{\tau=0}^{T-1} \beta^\tau c^\tau, \sum_{\tau=0}^{T} \beta^\tau c^\tau\right]$ in the beginning of the cycle. The shaded areas represent a density for the distribution of inventories conditional on the time index of the cycle τ.

As the cycle progresses, the consumer does not visit the seller and gradually depletes her inventory by consuming according to the sequence $\{c^\tau\}_{\tau=0,1,\ldots,T}$. This continues until her inventory at the cycle time $\tau = T$ is insufficient to meet her consumption needs c^T: $s \leq c^T$. At that point she buys a small top up bundle $\hat{x}(s) = c^T - s$ from the seller and consumes everything at her disposal. At the end of the cycle, the consumer has nothing left in her storage and she restarts the cycle by stockpiling.

The stationary consumer-optimal rules are not unique. There is an interval of optimal quantities that are purchased by a consumer without inventory:

$$\hat{x}(0) \in \left[\sum_{\tau=0}^{T-1} \beta^\tau c^\tau, \sum_{\tau=0}^{T} \beta^\tau c^\tau\right]$$

They choose the quantity in such a manner as to generate the equilibrium distribution of inventories. We characterize this distribution in the next section.

This purchasing behavior combined with our earlier point about deterministic consumption sequence $\{c^\tau\}_{\tau=0,1,\ldots,T}$ implies that on-path inventory for the consumer at the cycle time τ lies in the interval $S^\tau = \left[\underline{s}^\tau, \overline{s}^\tau\right]$, where

$$\begin{aligned}
\beta\underline{s}^\tau &= \underline{s}^{\tau-1} - c^{\tau-1}\\
\beta\overline{s}^\tau &= \overline{s}^{\tau-1} - c^{\tau-1}\\
\beta\underline{s}^1 &= \sum_{\tau=0}^{T-1} \beta^\tau c^\tau - c^*\\
\beta\overline{s}^1 &= \sum_{\tau=0}^{T} \beta^\tau c^\tau - c^*\\
\underline{s}^0 &= \overline{s}^0 = 0
\end{aligned}$$

Thus, the on-path consumer-optimal rules are

$$\forall s \in S^\tau, \tau \notin \{0, T\} : \hat{c}(s) = c^\tau,\ \hat{s}(s) = \beta^{-1}(s - c^\tau) \text{ and } \hat{x}(s) = 0. \tag{7}$$

The off-path part of the consumption rule can be defined using Proposition 2: if the consumer has inventory $s \in (\overline{s}^\tau, \underline{s}^{\tau-1})$, she consumes it in an optimal way and does not visit the sellers until she runs out. When her inventory is empty, she is back on equilibrium path.

There are two additional considerations. First, the cycles are not synchronized across the consumers. In the stationary equilibrium, the share of consumers that are consuming c^τ at a certain period is $1/(T+1)$. This way, the distribution of inventories in the population is time-invariant.

Second, this consumption rule (extended to the off path values of inventories) satisfies the condition (5) of Proposition 1. Therefore, we can find the menu pricing under which this consumption choice $\hat{c}$ together with the choice from the menu $\hat{x}$ is optimal.

5.2 Equilibrium prices

Now that we characterized optimal consumption, we can look for the menu that, on the one hand, induces this consumption, and on the other hand, is seller-optimal.

Consider a seller facing a population of consumers whose inventories are private and distributed according to some distribution F. Formally, this distribution is $F(z) = \mu(\{i : s_i \leq z, a_i = 1\})$, where μ is the measure defined on the set of consumers and s_i and a_i are measurable functions that captures consumers' inventories and shopping behavior in the current period.

Proposition 6. *There exists a stationary menu such that $\hat{c}(s)$ defined in equation* (7) *is consumer-optimal and maximizes seller's profits. Under this stationary menu:*

(i) The consumers shop only at the beginning and the end of their consumption cycle. The distribution F of inventories among consumers who attend the seller has the support $S^0 \cup S^T = [0, c^T]$.

(ii) The distribution F solves

$$F'(s)[u'(\hat{c}(s)) - k] = -F(s)u''(\hat{c}(s)). \tag{8}$$

Proof. To prove this proposition we first construct the stationary menu that induces $\hat{c}(\cdot)$. Then we look for the distribution of inventories under which such a menu is seller-optimal.

By Proposition 1, the menu that induces $\hat{c}$ must satisfy

$$p(\hat{x}(s)) = u(\hat{c}(s)) + \delta V(\hat{s}(s)) + \int_s^{c^T} u'(\hat{c}(z))dz - V(\bar{s}). \tag{9}$$

To show that, recall that $p(\hat{x}(s)) = u(\hat{c}(s)) + \delta V(\hat{s}(s)) - V(s)$. By integrating the envelope condition (4) we obtain

$$V(s) = V(c^T) - \int_s^{c^T} u'(\hat{c}(z))dz.$$

Combining this with the formula for the price yilds equation (9). Given the prices, we can calculate the seller's aggregate profit:

$$\pi = \int \left[u(\hat{c}(s)) + \delta V(\hat{s}(s)) - k\hat{x}(s) - V(c^T) + \int_s^{c^T} u'(\hat{c}(z))dz \right] dF(s).$$

The equation (8) is a first order condition for the programme that maximizes profit point-wise for each level of inventory s. Note that the profit extracted from a consumers with a given inventory s is concave in $\hat{c}$, therefore the first order condition is both necessary and sufficient for the optimality of the menu.

Note that

$$\hat{x}(s) = \begin{cases} x^0, & \text{if } s = 0, \\ c^T - s, & \text{if } s \in (0, c^T], \\ 0, & \text{otherwise.} \end{cases}$$

where $x^0 > c^T$ is different for different consumers. This implies that it is optimal for consumers with $s \geq c^T$ not to attend the seller. All consumers with $s < c^T$ strictly prefer to shop. What remains to show is that the constructed $\hat{x}(s)$ satisfies equation (5). Note that for any stationary menu, $\tilde{c}(r)$ is increasing and therefore $u'(\tilde{c}(r))$ is decreasing in r. Given this monotonicity, any decreasing function $\hat{x}(s)$ satisfies condition (5). □

The next step in our analysis is to construct the distribution of inventories F and link it to the length of the consumption cycle. Suppose that all the consumers whose consumption is c^0 and a fraction α of the consumers whose consumption is c^T in the current period attend the seller:

$$\mu(\{i : a_i = 1, c_i = c\}) = \begin{cases} \frac{1}{T+1}, \text{ if } c = c^0 \\ \frac{\alpha}{T+1}, \text{ if } c = c^T \\ 0, \text{ otherwise.} \end{cases}$$

Then, the share of the consumers who attend the shop with an empty storage is $F(0) = \frac{1}{1+\alpha}$ and every consumer at the shop has inventory below c^T, therefore $F(c^T) = 1$. We rewrite quation (8) as

$$F'(s) = F(s)\frac{-u''(c^T)}{u'(c^T) - k},$$

and integrate it to obtain

$$\log(1+\alpha) = \frac{-c^T u''(c^T)}{u'(c^T) - k}. \tag{10}$$

This gives us the share of the consumers $\alpha = 1 - e^{\frac{-c^T u''(c^T)}{u'(c^T)-k}}$.

The share α is crucial in finding the length of the consumer cycle.

Proposition 7. *Let $\mathcal{T}$ solve $c^{\mathcal{T}} u''(c^{\mathcal{T}}) + \left(u'(c^{\mathcal{T}}) - k\right)\log 2 = 0$. For any integer $T > \mathcal{T}$, there exists a stationary equilibrium with the consumption cycle of length T.*

Proof. Consider equation (10). Because $0 \le \alpha \le 1$, it must be the case that

$$\log 2 \ge \frac{-c^T u''(c^T)}{u'(c^T) - k} > 0.$$

There always exists T that satisfies the inequality since the right-hand side is arbitrarily close to zero when T gets arbitrarily large. Moreover, because $u'''(c) \le 0$ for all c, there exists a threshold $\mathcal{T}$ such that any $T > \mathcal{T}$, c^T satisfies (10). □

Note that the distribution F can be constructed only if some consumers with high level of inventory do not go shopping. Our modeling choice with additional decision $a_{i,t}$ allows for it. Of course, as we showed in the proof of Proposition 6, for these consumers not shopping is an optimal choice. The seller's menu is such that should a consumer who stays at home visit the seller, she would find the item $x = 0$ to be the most attractive given how much inventory she has.

The entire distribution of inventory (i.e, without conditioning on $a_{i,t} = 1$) has a support

$$\{0\} \cup [0, c^T] \cup [c^{T-1}, \beta c^T + c^{T-1}] \cup [\beta c^{T-1} + c^{T-2}, \beta^2 c^T + \beta c^{T-1} + c^{T-2}] \cup \ldots$$
$$\cup \left[\sum_{\tau=1}^{T-1} \beta^{\tau-1} c^\tau, \sum_{\tau=1}^{T} \beta^{\tau-1} c^\tau\right].$$

The distribution in these intervals are clones of each other scaled by β. On $[0, c^T]$ there are two mass points. One mass point of $\frac{1-\alpha}{T+1}$ at c^T and the other mass point of $1/(T+1)$ at 0. Formally, the distribution is

$$\mu(\{i : s_i < s\}) = \frac{1}{T+1}\begin{cases} 1, \text{ if } s = 0, \\ e^{\frac{-s u''(c^T)}{u'(c^T)-k}}, \text{ if } s \in (0, c^T), \\ 2, \text{ if } s = c^T. \end{cases} \tag{11}$$

Proposition 7 states that there are multiple stationary equilibria. If an equilibrium with a length of the cycle $T+1$ exists, one can construct an equilibrium for any $\hat{T} + 1 > T + 1$ as well. These equilibria are Pareto ordered because they share the first $T+1$ elements of the consumption stream. The multiplicity is caused by a coordination failure: If the sellers expect the consumers to shop less frequently, the equilibrium menus induce longer cycles and vice versa. Thus, it is natural to consider an equilibrium with the shortest cycle as the most relevant. For the rest of the analysis we focus on the equilibrium with the shortest cycle length.

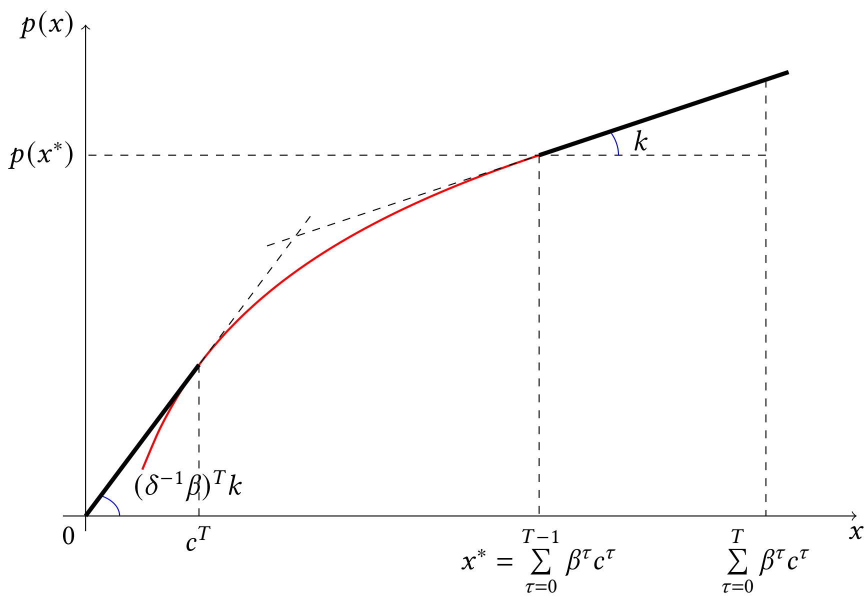


Fig. 2. Equilibrium menu.
Note: solid black lines represent items available on the menu; red line represents the indifference condition that relates prices in different parts of the menu to each other.

There are two features of the equilibria that help us identify prices. First, only consumers with inventory $s \in [0, c^T]$ shop. Second, these consumers either consume c^* (if $s = 0$) or c^T (if $s \in (0, c^T]$). Thus, the *marginal prices* at which these consumers buy product are $u'(c^T)$ for small items $x \in [0, c^T]$ and $u'(c^*)$ for large items.

When the consumer has no inventory, she has two optimal ways to proceed. She can either reset the consumption cycle by stockpiling or buy another top-up item and consume c^T for one more period. This indifference pins down the premium this consumer pays for large bundles. In particular, consider $x^* = \sum_{\tau=0}^{T-1} \beta^\tau c^\tau$ that costs p^*:

$$(1 - \delta)V(0) = u(c^T) - (\delta^{-1}\beta)^T k c^T$$
$$(1 - \delta^T)V(0) = \sum_{\tau=0}^{T-1} \delta^\tau u(c^\tau) - p^*.$$

Therefore,

$$p^* = \sum_{\tau=0}^{T-1} \delta^\tau u(c^\tau) - \frac{1-\delta^T}{1-\delta}\left(u(c^T) - (\delta^{-1}\beta)^T k c^T\right). \tag{12}$$

We summarize the description of the equilibrium menu in the following Proposition (also, see Figure 2).

Proposition 8. *Each seller t offers two types of items in a stationary equilibrium:*

(i) shop-to-mouth (or top-up) items are priced linearly with a constant markup: for any $x \leq c^T$:

$$p_t(x) = (\delta^{-1}\beta)^T kx;$$

(ii) items for stockpiling are priced via a two-part tariff: for any $x \geq \sum_{\tau=0}^{T} \beta^\tau c^\tau$:

$$p_t(x) = p^* + k(x - x^*);$$

(iii) no other items are available: for any $x \in \left(c^T, \sum_{\tau=0}^{T} \beta^\tau c^\tau\right)$: $p_t(x) = \infty$.

Proof. Since the consumption $\hat{c}(s)$ is piece-wise flat, the marginal price is constant—i.e., the menu is piece-wise linear. Moreover, the marginal price should be equal to the marginal utility of consumption, namely, k and $(\delta^{-1}\beta)^T k$ for the two segments. Finally, the intercept for the part of the menu that features larger bundles is characterized by (12). □

This menu is optimal in the space of all menus. However, given the linear cost of storage, it is instructive to look at this menu from the point of view of a simple model of price competition à la Bertrand.

Consider a single cohort of consumers—i.e., all the consumers who consume the same amount within a period. Across the consumption cycle, the good is sold to them by two sellers separated by T periods in time. Indeed, when the consumer buys a top-up bundle, some of the good is supplied by the current seller and the rest comes from consumer's inventory, which means that it was sold to her by another seller T periods ago.

These two sellers compete in *marginal prices* for the part of consumption stream that occurs at the end of consumption cycle:

(i) the two sellers cannot observe each others' menus when setting their own;
(ii) the ability to store allows the consumers to fine-tune how they procure their consumption stream: if a consumer buys one less marginal unit from the first seller (when stockpiling), she can buy one more marginal unit from the seller T periods later when topping up her consumption in that period to c^T.
(iii) The seller that operates in the last period of the consumption cycle has an advantage over the seller from T periods prior to that—the former procures the good at the marginal cost k, whereas the latter does so at the effective marginal cost $(\delta^{-1}\beta)^T k$ due to costly storage and discounting.

Such a competition results in marginal prices being driven to the largest of the two marginal costs, namely to $(\delta^{-1}\beta)^T k$. The seller that sells top ups makes profit on those since his marginal cost is k. At the same time a marginal unit sold for stockpiling brings zero marginal profit. However, the part of the menu that is meant for stockpiling still generates profit because the consumers cannot buy good for the first $T-1$ periods of the consumption cycle cheaply from any other seller. If the consumer has no inventory, the seller has market power when selling a good that is consumed immediately.

It is not surprising that large bundles meant for stockpiling appear in the equilibrium menu in the context of storable good[13] The presence of small bundles that are meant to be instantaneously consumed is less obvious, and it warrants a discussion. These bundles have an equilibrium nature—i.e., their presence cannot be explained using optimal pricing arguments alone.

These small, top-up bundles appear on the sellers' menus because consumers exhibit precautionary motives. If the seller were certain that consumers who show up at his store have no inventories, he would set high prices and exploit the fact that his customers have no alternative way to source the good. As a precaution, some consumers keep a small unpredictable amount of good in their storage towards the end of their consumption cycle. Thus, the seller has no choice but to screen such consumers from the ones that have no inventory by offering them small bundles.

5.3 Consumer surplus

Storage plays a central role in intertemporal arbitrage since it allows the consumer to avoid frequently buying small quantities at high per-unit price. It stirs up competition between sellers in different periods and helps to retain consumer surplus even when consumers are homogeneous in their tastes.

A surprising implication is that storage helps consumers obtain a certain amount of consumer surplus regardless of the cost of storage as long as consumers can choose the frequency of shopping. In particular, when storage is very costly, the consumers counter it by keeping smaller inventories and shopping more frequently. The sellers optimally adjust the prices in a way that keeps consumer surplus at a level that only depends on the shape of the utility function and not on the details of the storage technology or discounting.

To see this, consider the value of the consumer with an empty storage, i.e., $V(0)$. Let

$$\mathcal{U}(p) := (1-\delta)^{-1} \max_{c \geq 0} \{u(c) - pc\}$$

be the indirect utility of the consumer who can always buy the storable good at linear price p. This function is decreasing in p. Note, that for $p^T = (\delta^{-1}\beta)^T k$,

$$V(0) = (1-\delta)^{-1} \left(u(c^T) - (\delta^{-1}\beta)^T k c^T \right) = \mathcal{U}(p^T). \tag{13}$$

The following result summarizes our characterization of the consumer surplus.

Proposition 9. *In the equilibrium with the shortest consumption cycle, the consumer surplus of a consumer without inventory is*

$$V(0) = \mathcal{U}(p^T),$$

[13]For instance, the optimal menu found in Hendel et al. (2014) is designed to supply the consumer with ample amount of good eliminating the need for frequent restocking.

where[14]

$$p^T = k \exp\left(\left\lceil \frac{\log \tilde{p} - \log k}{\log(\delta^{-1}\beta)} \right\rceil \log(\delta^{-1}\beta)\right)$$

and $\tilde{p}$ solves

$$\log 2 = \frac{-u'^{-1}(\tilde{p})u''(u'^{-1}(\tilde{p}))}{\tilde{p} - k}. \tag{14}$$

Proof. This result follows directly from the characterization of the shortest equilibrium consumption cycle (see Proposition 7). □

This proposition makes two points. First, the equilibrium consumer surplus is same as if the consumers were presented with a stationary linear price p^T. This tells us that the ability to store the good allows the consumers to un-bundle items offered by sellers and linearize nonlinear pricing. Second, the price p^T is "close" to the price $\tilde{p}$ in the sense that the difference between the two is an artifact of modelling the problem in discrete time. Indeed, suppose the parameters of the model are such $p^T > \tilde{p}$. We can always split the period of our model into subperiods in such a way that

(i) the total storage cost (and discounting) per unit of time remains the same;
(ii) $p^T = \tilde{p}$ for the newly defined time periods.

Note that $p^T > \tilde{p}$ occurs because the consumers cannot fine-tune the frequency of shopping to their cost of storage. If the time grid is sufficiently fine, this issue does not arise. We illustrate this point on Figure 3 which shows the comparative statics of the price p^T with respect to the effective cost of storage $\delta^{-1}\beta$.

Thus, if we disregard the integer nature of shopping frequency, the consumer surplus is equal to $\mathcal{U}(\tilde{p})$. The consumer surplus is essentially independent of cost of storage and is determined entirely by the curvature of consumer's utility and marginal costs of production: the two parameters that guide optimal intertemporal decisions of the social planner. This result echoes the finding of Hendel et al. (2014) who show that the consumer surplus is independent of the consumers' storage capacity.

6 INFORMATION AND MARKET STRUCTURE

In our main model we assume that each seller is active for one period only and that a current seller does not observe past pricing. Both assumptions make significant contributions towards tractability—without them, in order to characterize equilibria one would need to keep track of a distribution of inventories in the population both on and off equilibrium path. Moreover, both on and off path menus should be optimal given these distribution and the evolution of the distribution should be consistent with the consumer optimal choice from these menus.

Solving such a model would be impractical. To shed light on the effects of various information and market structure we propose an alternative approach. We use an equilibrium in the two-period version of our model as a benchmark and explain how it changes when

(i) short-lived sellers can observe past menus; or when
(ii) there is a long-lived seller operating in both periods.

[14]The notation $\lceil x \rceil$ represents the smallest integer larger than x.

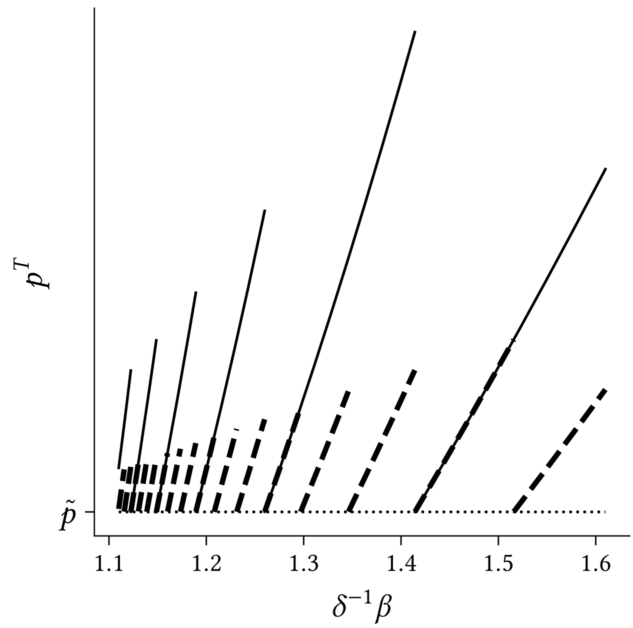


Fig. 3. Comparative statics of the price p^T with respect to the effective cost of storage $\delta^{-1}\beta$.
Note: Dotted line is the solution $\tilde{p}$ to equation (14); solid line is price p^T; dashed bold line is price p^T with the length of the time period shortened by the factor of 3.

We show that there is a close connection between our main model with infinite time horizon and the two-period benchmark. Thus, analyzing two-period model is helpful in developing intuition for the effects of information and presence of a long-lived seller in a more realistic setting with infinite time horizon.

We find that better information on past prices leads to higher prices. This occurs because sellers can credibly offer a lower variety of bundles. As a result, the consumers have less flexibility in terms of storing the good and, therefore, lower bargaining power. In addition to that, when the consumers shop repeatedly with the same seller, this long-lived seller creates deficit by offering smaller bundles than under serial monopoly. The seller does so to reduce cannibalization of the future demand by current sales.

6.1 Benchmark

This benchmark is the two-period version of our model that uses the same assumptions on information and market structure as our main model. In particular, the good is supplied by the serial monopoly and first-period pricing is not observed by the second-period seller. When we introduce new features into our analysis—e.g., a long-lived seller—we compare the resulting equilibrium with this benchmark.

The results from the main model carry over to this two-period version almost entirely. Propositions 1, 2, 3, 4 and 6 are valid in this setup. The only difference lies in the equilibrium menus and the distribution of the inventories. Because the stationarity condition does not apply in the two-period models, the first- and second-period menus are different and at the end of the second period every agent's storage is empty.

Nevertheless, conceptually, the menus have similar structure to the one characterized in Section 5.2. The first period menu offers a collection of items for stockpiling. It is implemented via the two part tariff with a per-unit price equal to the marginal cost of production. The smallest item available on the menu is c^* and it costs $p_1^* = u(c^*)$. Thus,

$$p_1(x) = \begin{cases} u(c^*) + k(x - c^*), & \text{if } x \geq c^*; \\ \infty, & \text{if } x < c^*. \end{cases}$$

In the second period, the seller offers the top-up items priced linearly with a positive markup. In addition to these, the seller offers and item c^* targeting consumers who did not store any good from the first period. This item is offered with a quantity discount compared to the smaller items. In particular,

$$p_1(x) = \begin{cases} \delta^{-1}\beta k x, & \text{if } x \leq c^1; \\ u(c^*) - u(c^1) + \delta^{-1}\beta k c^1 + k(x - c^*), & \text{if } x \geq c^*; \\ \infty, & \text{if } x \in (c^1, c^*). \end{cases}$$

The equilibrium menus are depicted on Figure 4. Just like in the main model, the choice of inventories in the first period is random and the distribution can be characterized using Proposition 6.

One can view the stationary menu from Section 5.2 as the first and the second period menus merged into one as long as the consumption cycle has a length $T + 1 = 2$. This means that the two period model directly compares to our main model only when the cost of storage is sufficiently large and $\mathcal{T} \leq 1$ (see Proposition 7). If $\mathcal{T} > 1$, we should expect some artifacts arising in the two-period model that are caused by the limited time-horizon.

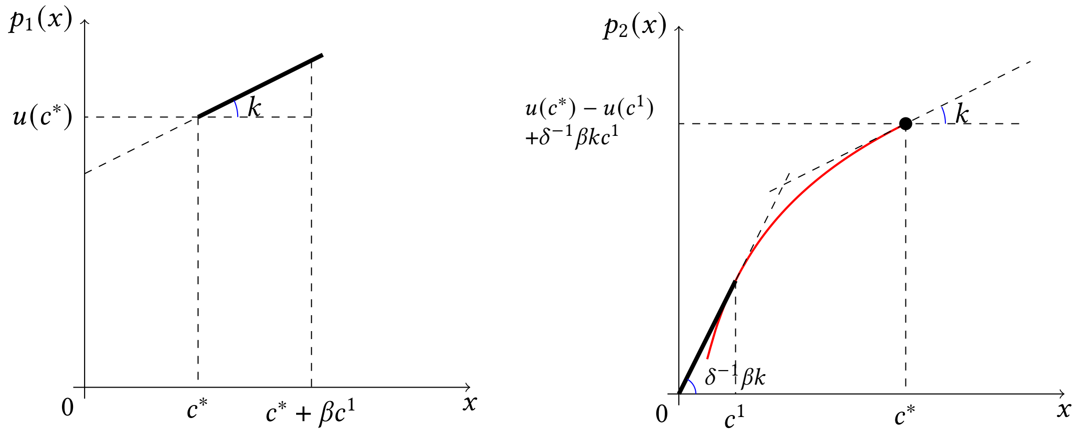
6.2 Observable past prices

In the market for storable good, concerns about consumer privacy arise even when the population is completely homogeneous in terms of taste. We showed that private inventories determine the willingness to pay, therefore sellers' access to any data that is informative of the distribution of inventories is bound to have an effect on prices and other equilibrium variables of interest.

To illustrate the effect of price data availability, we study a different information structure under which the sellers observe past pricing. Intuitively, if in the past the prices were high, it is natural to expect consumers to have fewer goods in their storage.

If the past pricing is publicly observed, it is optimal for the first seller to offer a single item on the menu to limit consumers' private information in the second period. Suppose the first seller offers x_1. The optimal second-period menu and consumer choices depend on x_1 as it is part of a public history. We use Proposition 2 to characterize the consumption stream following history x_1. Whenever $s > 0$,

$$\beta u'(x_1 - \beta s) = \delta u'(\hat{c}_2(s)). \tag{15}$$



(a) Menu at $t = 1$; (b) Menu at $t = 2$;

Fig. 4. Equilibrium menus.
Note: solid black lines represent items available on the menu; red line represents the indifference condition that relates prices in different parts of the menu to each other.

Let F be a distribution of consumer inventory s. Also, let $\bar{s}$ and $\underline{s}$ be the upper and the lower bounds of the support of F. Consumers with inventory $\bar{s}$ do not buy from the second seller—it follows from the seller's sequential rationality and Proposition 3. They consume $\bar{s}$ in the second period. In this case, equation (15) pins down $\bar{s}$:

$$\beta u'(x_1 - \beta\bar{s}) = \delta u'(\bar{s}).$$

The consumers with inventory $\underline{s}$ consume c^* (as long as $\underline{s} \leq c^*$, but this bound is not relevant on equilibrium path). Thus, the lower bound $\underline{s} = 0$ if $\beta u'(x_1) > \delta k$, and otherwise it solves

$$\beta u'(x_1 - \beta\underline{s}) = \delta k.$$

By Proposition 6,

$$F'(s)[u'(\hat{c}_2(s)) - k] = -F(s)u''(\hat{c}_2(s)), \tag{16}$$

with the boundary condition $F(\bar{s}) = 1$. The equations (15) and (16) together with the conditions for the bounds on inventories fully characterize equilibrium behavior in the second period given a first-period menu that features a single item x_1.[15]

Even though consumers diverge in their decisions, the value each get from buying x_1 from the seller is the same. Therefore, by selling x_1 in the first period, the seller collects profit

$$\pi_1(x_1) = u(x_1 - \beta\bar{s}) + \delta u(\bar{s}) - kx_1 - \delta V_2(0).$$

In this expression, both $\bar{s}$ and $V_2(0)$ depend on x_1.

[15]Recall that $\hat{c}_2(s), \underline{s}, \bar{s}, F(s)$ and $V_2(s)$ depend on x_1. For brevity, we omit x_1 from the list of the arguments of these functions.

The seller's maximization problem has an unusual property that both the inside and the outside options that consumer face are endogenous to x_1. For the value of the inside option, the dependence on x_1 is straightforward, but for the value of the outside option, it is not.

The menu offered to the consumers in the second period (and, therefore, the consumers' expectation of this menu) depends on the amount of good the consumers have in period 1, i.e., on x_1. The largest item on the second-period menu is purchased by the consumer with the lowest level of inventory. Note that opting out for the outside option in the first period would force consumer to have an empty inventory in the second period—in this case, the consumer will buy the largest item on the menu (usually, at a price below the value of this item). Thus, the value of the outside option in period 1 depends on the menu that is offered in this very period.

Because

$$V_2(0) = u(\bar{s}) + u(\hat{x}_2(\underline{s})) - u(\hat{c}_2(\underline{s})) - \int_{\underline{s}}^{\bar{s}} u'(\hat{c}_2(s))ds =$$

$$= u(\bar{s}) + u(\hat{x}_2(\underline{s})) - u(\hat{c}_2(\underline{s})) + \delta^{-1}[u(x_1 - \beta\bar{s}) - u(x_1 - \beta\underline{s})],$$

we obtain

$$\pi_1(x_1) = u(x_1 - \beta\underline{s}) - kx_1 + \delta[u(\hat{c}_2(\underline{s})) - u(\hat{x}_2(\underline{s}))]. \tag{17}$$

For any $x_1 \in [0, u'^{-1}(\delta\beta^{-1}k)]$, some consumers keep empty storage: $\underline{s} = 0$. When x_1 restricted to this interval, $\hat{c}_2(\underline{s}) = \hat{x}_2(\underline{s})$ and the profit is

$$\pi_1(x_1) = u(x_1) - kx_1.$$

It has a local maximum at $x_1 = c^*$. For $x_1 \in [u'^{-1}(\delta\beta^{-1}k), (1+\beta)c^*]$, the profit is increasing in x_1 and for all $x_1 > (1+\beta)c^*$, the profit is decreasing in x_1.

To summarize, it is optimal for the first seller to sell either $x_1 = c^*$ or $x_1 = (1+\beta)c^*$, depending on how efficient the consumer is at storing the good. If β is large enough, namely if,

$$\delta^{-1}\beta k \geq \frac{u(c^*)}{c^*},$$

the optimal menu in period 1 features $x_1 = c^*$ as a single item. From equation (17), price of this item is $p_1(c^*) = u(c^*)$.

When β is sufficiently small, the first seller uses a strategy that is not feasible for the models with infinite time horizon. In particular, the seller sells enough good to supply consumer for both periods and effectively exclude the second seller from the market. This is achieved by lowering consumers marginal utility in the second period beyond marginal cost of production. Clearly, when the time horizon is infinite, this strategy is not feasible as it would require the seller to sell infinite amount of good in the first period.

Just like in the benchmark setting, we focus on the case of large β. The first period seller sells $x_1 = c^*$ and the consumers' inventories range from $\underline{s} = 0$ to $\bar{s}$ that is strictly between zero and c^1.

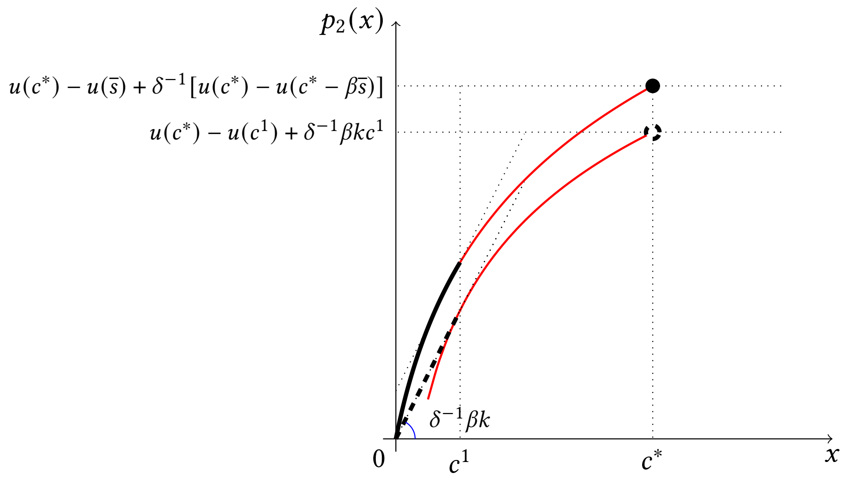


Fig. 5. Comparison of the benchmark menu and the public past prices menu.
Note: black lines represent items available on the benchmark menu (dashed) and the public past prices menu (solid); red line represents the indifference condition that relates prices in different parts of the menu to each other.

Proposition 10. *Suppose $\delta^{-1}\beta k \geq u(c^*)/c^*$. Compared to the benchmark, when past prices are public, there is an equilibrium in which:*

(i) the first period seller limits the menu to a singleton item $x_1 = c^*$ at the price $p_1 = u(c^*)$, and
(ii) the second period seller sets uniformly higher prices

$$p_2(x) = u(\tilde{s}(x) + x) - u(\bar{s}) + \delta^{-1}[u(c^* - \beta\tilde{s}(x)) - u(c^* - \beta\bar{s})],$$

where $\tilde{s}(x)$ solves $\beta u'(c^* - \beta\tilde{s}(x)) = \delta u'(x + \tilde{s}(x))$.

Proof. Given the induced consumption $\hat{c}_2(z)$, we can calculate the prices using equation (9):

$$\begin{aligned} p_2(\hat{x}_2(s)) =& u(\hat{c}_2(s)) - u(\bar{s}) + \int_s^{\bar{s}} u'(\hat{c}_2(z))dz = \\ & u(\hat{c}_2(s)) - u(\bar{s}) + \int_s^{\bar{s}} \delta^{-1}\beta u'(c^* - \beta z)dz = \\ & u(\hat{c}_2(s)) - u(\bar{s}) + \delta^{-1}[u(c^* - \beta s) - u(c^* - \beta\bar{s})]. \end{aligned}$$

Note that this menu is consistent with consumers choosing different inventories in the first period as the second period seller extracts any extra surplus above $u(x_1 - \beta\bar{s}) + \delta u(\bar{s})$ regardless of the choice of inventory s.

Also, note that $\hat{c}_2(s)$ is discontinuous at $s = 0$, therefore there is a gap in the menu. To consumers without inventory, the seller offers either c^* or $\lim_{s\to 0^+} \hat{c}_2(s) = c^1$. Items $x_2 \in (u'^{-1}(\delta^{-1}\beta k), c^*)$ are not offered.

For any $s > 0$, $p_2'(\hat{x}_2(s)) = \delta^{-1}\beta u'(c^* - \beta s) \geq \delta^{-1}\beta k$, which means that the prices are uniformly larger than in the case of hidden past menus. □

The comparison between the prices in the benchmark case and the case of public past prices is depicted on Figure 5. It is immediately clear from Proposition 10 that access to past prices reduces consumer surplus. In equilibrium, because the menu in the first period is more restrictive, fewer consumers stockpile which leads to higher future prices.

The equilibrium described in Proposition 10 is not unique. There are other equilibria in which the first seller offers so sell units of the good at the marginal cost in addition to the item $(c^*, u(c^*))$. These additional units do not affect the first seller's profit, but are beneficial to the consumer. Menus offered in the benchmark case are part of an equilibrium when past prices are observed.

6.3 Long-lived seller

In this section we consider the case of long-lived seller. Because of the perfect recall, the information structure in the model with the long-lived seller is the same as in the previous section. Most of the analysis carries over. The one feature that sets this case apart from the previous one is that the long-lived seller takes into account the effect of the first period menu on the second period profits. By selling too much good in the first period, he risks reducing the demand and profits in the second period. Thus, the seller offers a singleton menu in the first period (i.e., it is strictly suboptimal to offer additional units of the good at the per-unit price equal to the marginal cost), and, intuitively, we should expect the size of the item to be smaller than c^*.

The profit maximization problem is

$$\max_{x_1}\{\pi_1(x_1) + \delta\pi_2(x_1)\}$$

where $\pi_1(x_1)$ is characterized by (17) and

$$\pi_2(x_1) = \int_{\underline{s}}^{\bar{s}} \left[u(\hat{c}_2(s)) - k\hat{x}_2(s) - u(\bar{s}) + \int_{s}^{\bar{s}} u'(\hat{c}_2(z))dz \right] dF(s) =$$

$$-u(\bar{s}) + \left[u(c^*) - kc^* + \int_0^{\bar{s}} u'(\hat{c}_2(z))dz \right] F(0) +$$

$$\int_{\underline{s}}^{\bar{s}} \left[(u(\hat{c}_2(s)) - k\hat{x}_2(s))F'(s) + F(s)u'(\hat{c}_2(s)) \right] ds.$$

where $\hat{c}_2(s), \hat{x}_2(s), \underline{s}, \bar{s}$ and $F(s)$ all depend on x_1 in the way described in the previous section. Let x_1^m be a solution to the profit maximization problem.

Proposition 11. *Suppose $\delta^{-1}\beta k \geq u(c^*)/c^*$. The bundle sold by the long-lived monopolist in the first period is smaller than the analogous bundle sold by the short-lived monopolist: $x_1^m < c^*$*

Proof. For this result to hold, it suffices to show that $\pi_2(x_1)$ is decreasing. We prove this result in three steps. First, we show that the distribution of inventories is decreasing in x_1 in the first order stochastic dominance sense. Second, we show that in the optimal menu, the profit generated by a single consumer in the second period is decreasing in the consumer's inventory. Finally, we consider two scenarios when the seller sells x_1 or $\tilde{x}_1 < x_1$ in the first period. We argue that the second period menu that is optimal following the sale of x_1 generates higher profit when $\tilde{x}_1$ is sold (this immediately follows from the first two steps of the proof). Therefore, the second period menu that is optimal following the sale of $\tilde{x}_1$ generates higher profit than the one that is optimal following the sale of x_1.

To see that the distribution of inventories is decreasing in x_1 in the first order stochastic dominance sense, note that $\bar{s}$ is increasing in x_1 and, for a given s, $\hat{c}_2(s)$ is increasing in x_1. Also, because $-u''(c)/(u'(c) - k)$ is increasing in c (recall that $u'''(c) \leq 0$), $F'(s)$ is decreasing in x_1 for any s.

To see that the profit generated by a single consumer in the second period is decreasing in the consumer's inventory, recall that

$$p_2(\hat{x}_2(s)) - k\hat{x}_2(s) = u(\hat{c}_2(s)) - u(\bar{s}) + \delta^{-1}[u(x_1 - \beta s) - u(x_1 - \beta\bar{s})] - k(\hat{c}_2(s) - s),$$

therefore

$$\frac{d}{ds}[p_2(\hat{x}_2(s)) - k\hat{x}_2(s)] = (u'(\hat{c}_2(s)) - k)\frac{d\hat{c}_2(s)}{ds} - (\delta^{-1}\beta u'(x_1 - \beta s) - k) \leq 0.$$

□

This proposition illustrates a simple point we made in the beginning of this section. The long-lived monopolist faces the same problem as the sequence of the short-lived monopolists with one major difference: he is wary of cannibalizing the future market by overselling in the present and allowing the consumers to stockpile larger amounts of the good. To counter this, the long-lived monopolist creates deficit by selling relatively small bundles early on.

7 CONCLUDING REMARKS

This work is the first attempt to combine elements of competition and monopoly power in the market for nonlinearly priced storable goods. The seller in any period has monopoly power, but this is transitory, since he effectively competes with past and future sellers given the consumer's ability to store the good. Even though consumers are ex ante identical, they hold a distribution of inventories, giving rise to a nonlinear pricing problem in a dynamic context. Our model sheds light on important forces that shape equilibrium pricing, such as unavoidable emergence of consumer heterogeneity and inability of sellers to fully extract consumer surplus in bulk. Our model is complex enough that we have made several simplifying assumptions, such as linear storage costs costs and identical consumers. However, the model can be easily extended to the case of nonlinear cost of storage and consumer heterogeneity in cost or availability of storage. Our model and results may help in studying more complex phenomena such as equilibrium pricing by a long-lived seller, non-stationary price dynamics with intermittent sales, interaction between consumer taste and consumer inventories and selling goods through subscriptions.

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